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The Mathematics of Music and Art

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Music is a hidden arithmetic exercise of the soul,
which does not know that it is counting.
Gottfried Leibniz

To my wife, Helen, and my daughters, Ana and Mati. You are the rhythm, melody, and harmony in my life.

Preface

My grandparents owned a piano. As a child, every Sunday before having lunch with them, I would sit at the piano and play around with it, exploring. Very soon I started to notice patterns in melody and harmony; playing certain keys together would sound "good," others would not. Later I would learn about chords and simple progressions. These pattern-finding adventures would arise again and again as I started to become proficient and interested in mathematics. To this day, I enjoy both music and mathematics and continuously find that my interest in one subject feeds the other.

Stories about the relationship between music and mathematics are part of popular culture. How many times have we not heard that children exposed to classical music from an early age develop their analytical capacity, their geometric and mathematical skills, and that it has the effect of increasing the famous IQ? Conversely: the cases of famous music-loving scientists and mathematicians, such as Albert Einstein, Richard Courant, and many others, are notable. In this book, we want to explore more specifically and carefully the nature of this relationship, using mathematical tools to explain aspects of music theory.

Our efforts will take us through 2600 years of Western history, mathematics, science, art, and music. Pythagoras had his famous epiphany when he discovered the relationship between harmony and rational numbers: "All is number," he famously proclaimed. We will study this and other noteworthy aspects of this multifaceted relationship. This document is by no means definitive; the main purpose of it is not to document exhaustively but to try to transmit to the reader the enthusiasm that many have felt throughout history when they invented or discovered results, relationships, and concepts of extraordinary beauty, power, and simplicity.

Simple is the keyword here. The aim is to reach a wide audience. To do so, we must keep things light without trivializing the discussion. This is a delicate balance which we hope to maintain. If many readers can achieve small epiphanies of understanding and connecting, then the objective of this book will have been achieved.

It is suggested that all the proposed activities be carried out. Reading and accepting that a phenomenon occurs are not the same as witnessing it personally, and some of the activities try precisely to make the reader an eyewitness of these phenomena. Others are activities intended to reinforce newly presented concepts, ideas, and techniques. A webpage pointing to interactive activities for each chapter is available at https://themathofmusicandart.weebly.com/.

Everyone enjoys music. In fact, it shares with mathematics the nickname of universal language. We will see how, without mathematics, music as we know it could not have been developed. We also hope to exploit the relationship between music and mathematics to motivate and encourage the study of the queen of sciences with music. The language of music is deciphered by the language of mathematics.

This book is not all about music, and it's not all about math, but mainly about how the author, a professional mathematician and an amateur musician, views music through the eyes of mathematics. It is a very personal perspective.

It is also not really about the art of music. Art is in the details, in the particulars, not the generalities or universal patterns. Trained mathematicians with a scientific viewpoint look for generalities, patterns. These are undeniably important and fundamental to analyze and dissect music but are most likely not the source or the spark of beautiful songs and melodies. When one dissects, one kills; that is the price of understanding. Trying to use the ideas and concepts explored in this book to create new and beautiful art is a whole other matter which will not be explored in detail. The muses, creativity, will remain mysterious. I admit that when I sit down at the keyboard to play, it is tempting to try to turn off the rational and analytical side and tune in the emotional and sentimental. It is not easy to merge them both consciously. In art, great beauty implies structure, but structure does not imply beauty: just like knowing how to write a grammatically correct sentence does not imply breathtaking, powerful writing.

It should be noted that the focus of this document is Western or Eurocentric history and culture, which has been heavily influenced by African traditions and culture, particularly in music. Asian music developed quite independently and although there are many common points, this book will center on the Western or European experience. This is not a choice, but rather a result of the author's limitations. I hope that this personal and narrow view of the art form that most deeply affects and enchants humanity, music, will resonate with some, hopefully many.

These notes were written in preparation for and as part of an introductory interdisciplinary course: INTD3990 Music: Art, Science, and Mathematics, offered for the second time at the University of Puerto Rico at Mayagüez during the second semester of 2018–2019 in conjunction with two colleagues: Dana Collins, musicologist, and Héctor Jiménez, physicist. We thought about and planned the course for a couple of years; it had no prerequisites and was therefore open to a general audience, and consequently, it did not seek depth in any particular topic. The idea was to give the students a broad vision and achieve the integration of different and apparently disconnected subjects, with the aim of rounding out their general culture and education. We did, however, touch upon serious mathematics, music, and physics, which was presented in an informal and intuitive manner. Again, it was our hope that the material would be accessible to a wide audience.

Mayagüez, Puerto Rico
2023

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Introduction: Let's Start at the Beginning…

¿How old are music and mathematics? ¿When did we begin to create and understand these two disciplines? It turns out that there is archeological evidence that these two disciplines are as old as humanity. We have been counting and thinking about numbers, and making music, since the dawn of times.

Let's start with music. One of the oldest known instruments is a cave bear femur bone flute found in Slovenia, called the Divje flute. Its age is estimated at 50,000 to 60,000 years (Fig. 1).

Fig. 1 Divje flute, attribution: dalbera from Paris, France, CC BY 2.0 https://creativecommons.org/licenses/by/2.0, via Wikimedia Commons, Page URL: https://commons.wikimedia.org/wiki/File:Fl%C3%BBte_pal%C3%A9olithique_(mus%C3%A9e_national_de_Slov%C3%A9nie,_Ljubljana)_(9420310527).jpg

Fig. 2 Geisenklösterle cave flute, attribution: José-Manuel Benito Álvarez, CC BY-SA 2.5 https://creativecommons.org/licenses/by-sa/2.5, via Wikimedia Commons, Page URL: https://commons.wikimedia.org/wiki/File:Flauta_paleol%C3%ADtica.jpg

Other bone flutes have been found in Germany (Geisenklösterle Cave flutes) with an estimated age of 35,000 to 43,000 years (Fig. 2).

Let's note that the origins of agriculture and the first large settlements or cities it allowed are no more than 12,000 years old. So, these instruments are the work of our nomad, cave dwelling, very primitive ancestors. And it is no coincidence that these first instruments are carved out of bones, the same bones that our ancestors fed on, sucked the marrow out of to get the fat and nutrients that allowed them to grow large brains, scavenging them from the leftovers of the great animal hunters that roamed the land. It is both humbling and uplifting to think of these very primitive beings, scavenging and feeding from these bones and scraps, and yet having the mindfulness of recognizing their musical potential and the initiative to build toys, instruments, to play music, to enjoy. It says a lot about our species, and it fills me with wonder, optimism, and amazement to see ourselves as these very curious, smart, creative and playful creatures, right from our humble beginnings.

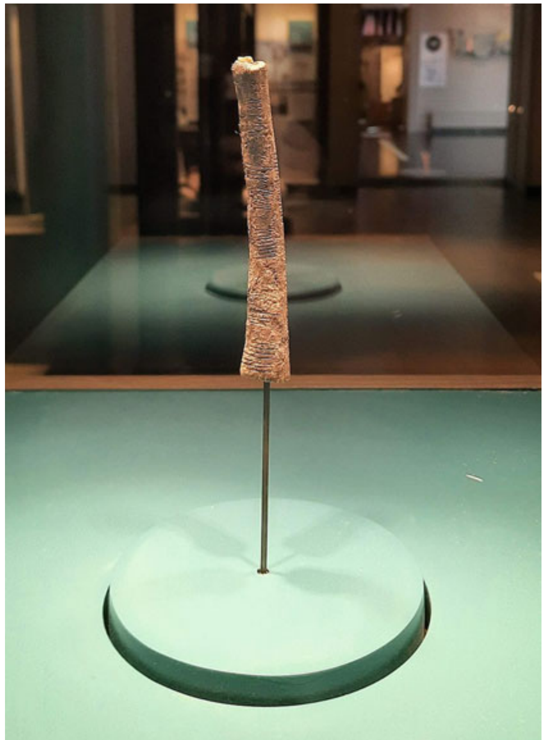
There is also another detail: bone artifacts last. Even earlier instruments made from wood or leather, for example, may have disappeared. But it is not hard to imagine our ancestors, sitting around the fire at night, clapping, stomping, hitting objects, percussions, to keep a steady beat while they sang stories, stories of the hunt, of their ancestors, of the heavens and the gods. I would imagine that percussions preceded these ancient flutes. And let's remember that our own bodies are instruments, our beating hearts, our voices, our whistling mouths, our palms, and feet. Language and music are one and the same, when we speak, we sing. The tone of our voice says as much as the words we speak. So, it is natural to speculate that music is much older than these already ancient artifacts (Fig. 3).

Now let's talk about mathematics. There are two notable archeological findings, the Lebombo bone and the Ishango bone, ages estimated at 43,000 years old and 20,000 to 22,000 years old, respectively, which are notched, as if keeping count. Some speculate that

Fig. 3 Sitting around the fire, attribution: Hynek Janáč, CC0, via Wikimedia Commons, Page URL: https://commons.wikimedia.org/wiki/File:People_sitting_around_a_camp_fire.jpg

the notches represent days in a lunar phase counting calendar, there are also speculations regarding the understanding of prime numbers, given groupings in the notches. In any case, we see clear evidence of humans counting, keeping records of said counts, and even understanding the divisibility properties of natural numbers. All of this happened tens of thousands of years ago. And interestingly, the artifacts are bone artifacts again, these ancestral bones that fed our bodies and our imaginations (Fig. 4).

Let's now speculate about the relationship between these findings. Can we make music without counting? Can we keep a beat, divide time, without mathematics, without numbers? In fact, the answer to these questions is an emphatic no. Even the simplest chanting, accompanied by a steady beat, implies keeping time; counting, preserving certain patterns that allow for prediction and preparation of what's to come, are absolutely necessary for practical and aesthetic reasons. So, the implication is that these two disciplines, music, and mathematics, must have initiated and developed hand in hand. As we shall see, this has continued to be the case throughout human history.



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Activity 1
Watch the videos on the activities page,[1] which talk about these ancient bone artifacts that represent humankind's beginnings in music and mathematics.

Activity 2
Visit the last link on the Additional Supplementary Materials section in the activities webpage (See Footnote 1), a webpage published by the American Mathematical Society with resources related to mathematics and music. Watch the PBS video titled *Majesty of Music and Math* (the first video you will find there), which will give you a truly wonderful introduction to topics we will discuss in the book.

[1] The activities and explorations can be found at https://themathofmusicandart.weebly.com/.

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