DATE

ADVANCED ELECTRODYNAMICS HW 3

① a) $\vec{E}(x,y,z,t) = E_0 \cos(kx - \omega t)\hat{y}$

$\vec{B}(x,y,z,t) = \frac{E_0}{c} \cos(kx - \omega t)\hat{z}$

WITH:

$\delta = 0$ $B_0 = \frac{E_0}{c}$

$k = \frac{\omega}{c}$

THE WAVE IS POLARISED IN THE y-DIRECTION SINCE $\vec{E} = E\hat{y}$

b) USING THE TRANSFORMATION RULES FOR $\vec{E}$ AND $\vec{B}$:

$E_x' = E_x \rightarrow E_x' = 0$

$E_y' = \gamma(E_y - vB_z) = \gamma E_0 [\cos(kx - \omega t) - \frac{v}{c}\cos(kx - \omega t)]$
$= \gamma E_0 \cos(kx - \omega t)[1 - \frac{v}{c}] = \alpha E_0 \cos(kx - \omega t)$

let $\alpha = \gamma(1 - \frac{v}{c})$

$E_z' = \gamma(E_z + vB_y) = 0$ SINCE $E_z = 0$, $B_y = 0$

$B_x' = B_x \rightarrow B_x' = 0$

$B_y' = \gamma(B_y + \frac{v}{c} E_z) = 0$ SINCE $B_y = 0$, $E_z = 0$

$B_z' = \gamma(B_z - \frac{v}{c^2} E_y) = \gamma E_0 [\frac{1}{c}\cos(kx - \omega t) - \frac{v}{c^2}\cos(kx - \omega t)]$

USING THE INVERSE LORENTZ TRANSFORMATIONS:

$x = \gamma(x' + vt')$
$t = \gamma(t' + \frac{v}{c^2}x')$

$kx - \omega t = k\gamma(x' + vt') - \omega\gamma(t' + \frac{v}{c^2}x')$ $k = \frac{\omega}{c}$
$= \gamma[(k - \omega\frac{v}{c^2})x' - (\omega - kv)t'] = k'x' - \omega' t'$

where:

$k' = (k - \omega\frac{v}{c^2})\gamma = k\gamma(1 - \frac{v}{c}) = \alpha k$

$\omega' = (\omega - kv)\gamma = \gamma\omega(1 - \frac{v}{c}) = \alpha\omega$

THE FIELDS ARE:

$\vec{E}'(x', y', z', t') = E_0' \cos(k'x' - \omega' t')\hat{y}$

$\vec{B}'(x', y', z', t') = \frac{E_0'}{c} \cos(k'x' - \omega' t')\hat{z}$

$E_0' = \alpha E_0$

c) $\cdot\ \omega' = \alpha\omega = \omega\gamma\left(1-\frac{v}{c}\right) = \omega\dfrac{\left(1-\frac{v}{c}\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \omega\sqrt{\dfrac{\left(1-\frac{v}{c}\right)^2}{\left(1-\frac{v}{c}\right)\left(1+\frac{v}{c}\right)}}$

$$\omega' = \omega\sqrt{\frac{1-v/c}{1+v/c}}$$

$\cdot\ \lambda' = \dfrac{2\pi}{k'} = \dfrac{2\pi}{\alpha k} = \dfrac{\lambda}{\alpha}$

$$\lambda' = \lambda\sqrt{\frac{1+v/c}{1-v/c}}$$

$\cdot\ \nu' = \dfrac{\omega'}{2\pi}\cdot\lambda' = \dfrac{\alpha\omega}{2\pi}\cdot\dfrac{\lambda}{\alpha} = \dfrac{\omega}{2\pi}\cdot\lambda = \nu\lambda = c$

d) $\dfrac{I'}{I} = \dfrac{E_0'^2}{E_0^2} = \alpha^2\dfrac{E_0^2}{E_0^2} = \alpha^2 = \dfrac{1-\frac{v}{c}}{1+\frac{v}{c}}$

IF $v \to c$ $\quad \alpha \to 0$ AND $E_0', \omega', I' \to 0$, $\lambda' \to \infty$

DATE

(2)

a) $A^\mu = (0, 0, A_y, 0)$ is the four-potential

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{\partial A^\nu}{\partial \mu} - \frac{\partial A^\mu}{\partial \nu}$ is the tensor field

where $\mu, \nu = 0, 1, 2, 3$

only $A^2 = A_y \neq 0$

$F^{02} = \partial^0 A^2 = \frac{E_y}{c} \rightarrow c E_y = \frac{\partial A_y}{\partial t}$ $\quad F^{20} = -F^{02}$

$F^{23} = -\partial^3 A^2 = B_x \rightarrow B_x = -\frac{\partial A_y}{\partial z}$ $\quad F^{32} = -F^{23}$

$F^{12} = \partial^1 A^2 = B_z \rightarrow B_z \frac{\partial A_y}{\partial x}$ $\quad F^{21} = -F^{12}$

- we can obtain the fields from a suitable A^μ

DATE

IN TENSOR NOTATION:

3

a) MAXWELL EQUATION: $\partial_\nu F^{\mu\nu} = \mu_0 J^\mu$

CONTINUITY EQUATION $\partial_\mu J^\mu = 0$ where $\partial_\mu J^\mu$ is THE 4-DIMENSIONAL DIVERGENCE OF J^μ

DIFFERENTIATING MAXWELL'S EQ:

$\partial_\mu \partial_\nu F^{\mu\nu} = \mu_0 \partial_\mu J^\mu$

$F^{\mu\nu} = -F^{\nu}{}_{\mu}$ THE TENSOR IS ANTISYMMETRIC

$\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$ DIFFERENTIATION IS SYMMETRIC

$\Rightarrow \partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu \partial_\mu F^{\nu\mu} = \partial_\mu \partial_\nu (-F^{\mu\nu}) = -\partial_\mu \partial_\nu F^{\mu\nu}$
(μ AND ν ARE DUMMY VARIABLES)

THEN:
$\partial_\mu \partial_\nu F^{\mu\nu} = -\partial_\mu \partial_\nu F^{\mu\nu} \Rightarrow \partial_\mu \partial_\nu F^{\mu\nu} = 0$
PUTTING THIS IN THE MAXWELL EQUATION:

$\partial_\mu \partial_\nu F^{\mu\nu} = \mu_0 \partial_\mu J^\mu = 0 \Rightarrow \boxed{\partial_\mu J^\mu = 0}$ CONTINUITY EQUATION

b) MAXWELL'S EQUATION:
$\partial_\nu G^{\mu\nu} = 0$

- $\partial_\nu G^{\mu\nu} = 0$ IS EQUIVALENT TO $\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \end{cases}$

- CONSIDER:
$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$

THE INDEX μ, ν, λ CAN TAKE THE VALUES: 0, 1, 2, 3

CASE 1: μ, ν, λ ALL SPATIAL

IF $\mu = 1$, $\nu = 2$, $\lambda = 3$

$\partial_3 F_{12} + \partial_1 F_{23} + \partial_2 F_{31} = 0$ SINCE $F_{12} = B_z$, $F_{23} = B_x$, $F_{31} = B_y$

$\dfrac{\partial B_z}{\partial z} + \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} = \vec{\nabla} \cdot \vec{B} = 0$ IT IS TRUE FOR THE PERMUTATION OF THE INDICES

CASE 2: μ, ν, λ ONE TEMPORAL, 2 SPATIAL

IF $\mu = 0$, $\nu = 1$, $\lambda = 2$

$\partial_2 F_{01} + \partial_0 F_{12} + \partial_1 F_{20} = 0$ $F_{01} = -\dfrac{E_x}{c}$, $F_{20} = \dfrac{E_y}{c}$, $F_{12} = B_z$

$\dfrac{\partial}{\partial y}\left(-\dfrac{E_x}{c}\right) + \dfrac{\partial B}{\partial (ct)} + \dfrac{\partial}{\partial x}\left(\dfrac{E_y}{c}\right) = 0$

$\Rightarrow \dfrac{\partial B_z}{\partial t} = -\left[\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y}\right]$ THIS IS THE Z-COMPONENT OF $-\dfrac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$

- PUTTING
$\mu = 0 \quad \nu = 2,3 \quad \lambda = 1,3$ WE FIND THE OTHER COMPONENTS
THEN
$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ IS EQUIVALENT TO $\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \end{cases}$
AND WE CONCLUDE THAT IT IS ALSO EQUIVALENT TO $\partial_\nu G^{\mu\nu} = 0$

a) FIELD TENSORS $F^{\mu\nu}$ AND $G^{\mu\nu}$

$$F^{\mu\nu} = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix} \qquad G^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix}$$

LOWERING THE INDEX MAKES THE TENSOR COVARIANT, CHANGE THE SIGN OF THE ZEROTH COMPONENT

- $F_{\mu\nu}F^{\mu\nu} = F_{00}F^{00} + F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + F_{10}F^{10} + F_{11}F^{11} + F_{12}F^{12} + F_{13}F^{13} + F_{20}F^{20} + F_{21}F^{21} + F_{22}F^{22} + F_{23}F^{23} + F_{30}F^{30} + F_{31}F^{31} + F_{32}F^{32} + F_{33}F^{33}$
$= (-)(-)F^{00}F^{00} - F^{01}F^{01} - F^{02}F^{02} - F^{03}F^{03} - F^{10}F^{10} + F^{11}F^{11} + F^{12}F^{12} + F^{13}F^{13} + F^{20}F^{20} + F^{21}F^{21} + F^{22}F^{22} + F^{23}F^{23} + F^{30}F^{30} + F^{31}F^{31} + F^{32}F^{32} + F^{33}F^{33} =$

$= (F^{00})^2 - (F^{01})^2 - (F^{02})^2 - (F^{03})^2 - (F^{10})^2 + (F^{11})^2 + (F^{12})^2 + (F^{13})^2 - (F^{20})^2 + (F^{21})^2 + (F^{22})^2 + (F^{23})^2 + (F^{30})^2 + (F^{31})^2 + (F^{32})^2 + (F^{33})^2 =$
$= 0 + \left(\frac{E_x}{c}\right)^2 - \left(\frac{E_y}{c}\right)^2 - \left(\frac{E_z}{c}\right)^2 - \left(-\frac{E_x}{c}\right)^2 + 0 + B_z^2 + B_y^2 - \left(-\frac{E_y}{c}\right)^2 + B_z^2 + 0 + B_x^2 - \left(\frac{E_z}{c}\right)^2 + B_y^2 + B_x^2 + 0$

$F_{\mu\nu}F^{\mu\nu} = -2\left(\frac{E_x}{c}\right)^2 - 2\left(\frac{E_y}{c}\right)^2 - 2\left(\frac{E_z}{c}\right)^2 + 2B_x^2 + 2B_y^2 + 2B_z^2$
$= -\frac{2}{c^2}(E_x^2 + E_y^2 + E_z^2) + 2(B_x^2 + B_y^2 + B_z^2)$

AND: $\boxed{F_{\mu\nu}F^{\mu\nu} = 2\left(B^2 - \frac{E^2}{c^2}\right)}$

$\boxed{G_{\mu\nu}G^{\mu\nu} = 2\left(\frac{E^2}{c^2} - B^2\right)}$ SINCE $G^{\mu\nu}$ CAN BE OBTAINED FROM $F_{\mu\nu}$ WITH THE SUBSTITUTION $\frac{E_x}{c} \rightleftarrows B_x$, $\frac{E_y}{c} \rightleftarrows B_y$, $\frac{E_z}{c} \rightleftarrows B_z$

$F_{\mu\nu}G^{\mu\nu} = \cancel{F_{00}F^{00}} - F^{01}G^{01} - F^{02}G^{02} - F^{03}G^{03} - F^{10}G^{10} + \cancel{F^{11}}G^{11} + F^{12}G^{12} + F^{13}G^{13} - F^{20}G^{20} + F^{21}G^{21} + \cancel{F^{22}}G^{22} + F^{23}G^{23} - F^{30}G^{30} + F^{31}G^{31} + F^{32}G^{32} + \cancel{F^{33}}G^{33} =$

$= -F^{01}G^{01} - F^{02}G^{02} - F^{03}G^{03} - [(-F^{01})(-G^{01})] + F^{12}G^{12} + F^{13}G^{13} - [(-F^{02})(-G^{02})] + [(-F^{12})(-G^{12})] + F^{23}G^{23} - [(-F^{03})(-G^{03})] + [(-F^{13})(-G^{13})] + [(-F^{23})(-G^{23})] =$

$$= -2\left(F^{01}G^{01} + F^{02}G^{02} + F^{03}G^{03}\right) + 2\left(F^{12}G^{12} + F^{13}G^{13} + F^{23}G^{23}\right) =$$

$$= -2\left[\frac{E_x}{c}\cdot B_x + \frac{E_y}{c}B_y + \frac{E_z}{c}B_z\right] - 2\left[B_z\frac{E_z}{c} + B_y\frac{E_y}{c} + B_x\frac{E_x}{c}\right] =$$

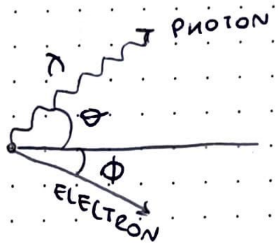
$$= -\frac{2}{c}(\vec{E}\cdot\vec{B}) - \frac{2}{c}(\vec{E}\cdot\vec{B})$$

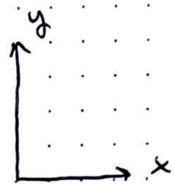
$$\boxed{F_{\mu\nu}G^{\mu\nu} = -\frac{4}{c}(\vec{E}\cdot\vec{B})}$$

b) no, because the scalar invariant $F^{\mu\nu}F_{\mu\nu} = 2\left(B^2 - \frac{E^2}{c^2}\right)$ must be the same in both frames.
For a purely electric field $B=0$ and $F^{\mu\nu}F_{\mu\nu}$ is negative, for a purely magnetic field $E=0$ and $F^{\mu\nu}F_{\mu\nu}$ will be positive: this is not possible!

9) COMPTON EFFECT :

BEFORE





THE MOMENTUM 4-VECTOR is p^μ:

$$p^\mu = (E, p_x c, p_y c, p_z c)$$

in this problem we have to apply the conservation of energy and momentum, that is:

$$p^\mu_{BEFORE} = p^\mu_{AFTER}$$

a) BEFORE THE COLLISION

$$p^\mu_\gamma = \left(\frac{hc}{\lambda_0}, \frac{hc}{\lambda_0}, 0, 0\right) \quad , \quad p^\mu_e = (mc^2, 0, 0, 0)$$

(PHOTON) (ELECTRON)

b) AFTER THE COLLISION ;

$$p^{\mu'}_\gamma = \left(\frac{hc}{\lambda}, \frac{hc}{\lambda}\cos\theta, \frac{hc}{\lambda}\sin\theta, 0\right) , \quad p_e^{\mu'}$$

the conservation of energy and momentum is

$$p^\mu_\gamma + p^\mu_e = p^{\mu'}_\gamma + p_e^{\mu'}$$

$$(p^\mu_\gamma\, p_e^\mu - p_\gamma^{\mu'}) = (p_e^{\mu'})^2$$

$$(p_\gamma^\mu)^2 + (p_e^\mu)^2 + (p_\gamma^{\mu'})^2 + 2p_e^\mu(p_\gamma^\mu - p_\gamma^{\mu'}) - 2p_\gamma^\mu p_\gamma^{\mu'} = (p_e^{\mu'})^2$$

$$0 + m^2c^4 + 0 + 2mc^2\left(\frac{hc}{\lambda_0} - \frac{hc}{\lambda}\right) - 2\frac{hc}{\lambda_0}\frac{hc}{\lambda}(1-\cos\theta) = m^2c^4$$

SINCE $(p^\mu)^2 = E^2 - |\vec{p}|^2c^2 = m^2c^4$

$$\Rightarrow mc^2\left(\frac{hc}{\lambda_0} - \frac{hc}{\lambda}\right) = \frac{hc}{\lambda_0}\frac{hc}{\lambda}(1-\cos\theta) \Rightarrow \boxed{\lambda = \lambda_0 + \frac{h}{mc}(1-\cos\theta)}$$

and the energy of the emitted photon is: $E = \frac{hc}{\lambda}$

(5) $\gamma + e^- \rightarrow e^- + e^+ + e^-$

it is an inelastic collision since the particles after the collision are different than before.
let p the 4-momentum of the particles.
The energy and momentum conservation is:

$$p_1 + p_2 = p_3 + p_4 + p_5$$

we HAVE:

$$p_1 = \left(\frac{hc}{\lambda_0}, \frac{hc}{\lambda_0}, 0, 0\right), \quad p_2 = (mc^2, 0, 0, 0)$$

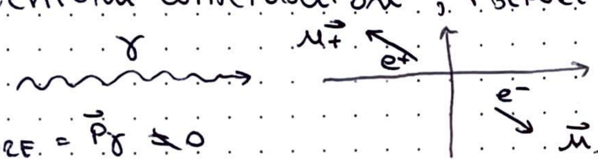
$S = (p_1 + p_2)^2$ IS INVARIANT AND MUST BE:

$$S \geq (mc^2 + mc^2 + mc^2)^2$$

The minimum energy required is $E_{MIN} = \sqrt{S_{MIN}} = 3mc^2$
$(m = m_e)$
the minimum energy of the photon: $E = \frac{hc}{\lambda} = 2mc^2$

- $\gamma \rightarrow e^+ + e^-$

this reaction is not possible because it is violated the momentum conservation, $\vec{P}_{BEFORE} = \vec{P}_{AFTER}$



$\vec{\mu}_- = -\vec{\mu}_+$

$\vec{P}_{BEFORE} = \vec{P}_\gamma \neq 0$

$\vec{P}_{AFTER} = \vec{P}_+ + \vec{P}_- = 0$

$\vec{P}_{BEFORE} \neq \vec{P}_{AFTER} \rightarrow$ the reaction is impossible.