Practice session: Physics of Stars

Kapteyn Learning Community Sirius A
Give motivations and/or derivations for your answers.

Calculator is allowed. Closed book. No loud chewing.

1. **Basic Principles**

(a) Briefly describe the following terms and, if applicable, give their units in cgs:
   1. Specific intensity I_ν
   2. Average intensity J_ν
   3. Monochromatic Flux F_ν
   4. Eddington Flux H_ν
   5. Effective Temperature
   6. Local Thermodynamic Equilibrium

(b) Briefly describe the difference between the Saha equation and the Boltzmann equation.

(c) Name the below equation and explain each variable.

$$\frac{n_{II}}{n_I} = \frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\frac{2U_{II}}{U_I}e^{-E_{ion}/kT}$$

(d) Use this equation to calculate the number density of ionised hydrogen at a depth in a star where $T = 17000$ K. Assume that 30 percent of the electrons are from the ionisation of hydrogen (i.e., $0.3n_e = n_{II}$) and that 90 percent of the hydrogen is ionized. Also assume $U_I = 2$.

Solution:

(a)
1. Specific intensity I_ν: Defines the directional value of the radiation field. Its units are erg/s/Hz/cm^2/sr.
2. Average intensity J_ν: The average of the specific intensity over all solid angles (directions). Its units are the same as those of the specific intensity.
3. Monochromatic Flux F_ν: The result of integrating the specific intensity over all solid angles (directions). Its units are erg/s/Hz/cm^2. It represents the energy received by an observer per units time per unit wavelength per area.
4. Eddington Flux H_ν: First moment of the specific intensity. Related to the flux through $\frac{F_\nu}{4\pi} = H_\nu$. Units of erg/s/Hz/cm^2.
5. Effective Temperature: The temperature needed for a blackbody to radiate the same amount of energy per unit surface area as the star. Units of Kelvin.
6. LTE comes in three forms. The first and weakest form simply states that the atomic energy levels and ionic populations are determined only by collisions in the plasma. A more stringent definition of the LTE is that the radiation emitted by matter is such that the source function is equal to a black body. A third, and even stricter, definition is that the specific intensity must be equal to a blackbody.

(b) The difference between the Saha and the Boltzmann equation is that the former describes the ionization states of the atom, whereas the latter describes excitation states.

(c) This is the Saha equation. $\frac{n_{II}}{n_I}$ is the ratio of number densities for two ionization states. n_e is the electron number density. m_e is the electron mass. U_I and U_{II} are the partition functions and E_{ion} is the ionization energy of the atom in the lowest ionisation state.

(d) Since we are working with ionization states, we use the Saha equation:

$$\frac{n_{II}}{n_I} = \frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\frac{2U_{II}}{U_I}e^{-E_{ion}/kT} \tag{1}$$

Furthermore, we are given that $T = 17000K$, that 30% of the electron number density comes from ionized hydrogen ($0.3n_e = n_{II}$) and that 90% of the hydrogen is ionized ($\frac{n_{II}}{n_{II}+n_I} = 0.9$). From the last assumption we get $n_I = \frac{1}{9}n_{II}$. Finally, we know that $U_I = 2$, $U_{II} = 1$ and $E_{ion} = 13.6$ eV. Now we can rewrite the Saha equation to

$$\frac{n_{II}}{\frac{1}{9}n_{II}} \cdot \frac{1}{0.3}n_{II} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\frac{2}{2}e^{-E_{ion}/kT} \tag{2}$$

After simplifying the left-hand side and filling in all the numbers, we get that

$$n_{II} = 1.66 \cdot 10^{16}\ \text{cm}^{-3} \tag{3}$$

2. **Variable Stars**

(a) Briefly describe what a variable star is and give some examples.

(b) What field of astronomy uses variable stars to study stellar interiors? What kind of observational data is needed for this?

Solution:

(a) A variable star is a star which has a periodic change in luminosity. Examples include Cepheids, RR Lyrae stars, δ Scuti stars and ZZ Ceti stars.

(b) Asteroseismology. You need apparent magnitude measurements over multiple epochs to do this.

3. **Hydrostatics**

(a) Derive the virial theorem for a star in hydrostatic equilibrium.
You may find the following equations useful;
The equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{\rho(r)GM(r)}{r^2}, \tag{4}$$

and the equation of potential energy

$$\Omega = -\int_0^{M_*} \frac{GM(r)}{r} dM. \tag{5}$$

Also, the thermal energy density ε in units of erg/cm^3 is given by: $\varepsilon = \frac{N}{V}\frac{3}{2}kT$.

Solution:

(a) Starting with the equation for hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{\rho(r)GM(r)}{r^2}. \tag{6}$$

Taking the dr to the other side and multiplying with the volume, we get

$$\frac{4}{3}\pi r^3 dP = -\frac{4}{3}\pi r\rho(r)GM(r)dr. \tag{7}$$

Also, we have the mass of a shell of width dr at a radius r:

$$dM(r) = 4\pi r^2\rho(r)dr, \tag{8}$$

which turns the previous equation into

$$VdP = -\frac{GM(r)}{3r}dM(r). \tag{9}$$

This equation can be integrated over the entire star. First, the left-hand side can be integrated by parts:

$$\int_{P_{centre}}^{0} VdP = PV|_{centre}^{surface} - \int_0^{V_*} PdV, \tag{10}$$

where V_* is the total volume of the star. Since the pressure at the surface is zero and the volume in the centre as well, the first term on the right-hand side is just zero. Then we can write

$$\int_0^{V_*} PdV = \frac{1}{3}\int_0^{M_*} \frac{GM(r)}{r}dM. \tag{11}$$

The potential energy can be written as

$$\Omega = -\int_0^{M_*} \frac{GM(r)}{r}dM, \tag{12}$$

so we now have

$$3\int_0^{V_*} PdV = -\Omega. \tag{13}$$

We are given that $\epsilon = \frac{N}{V}\frac{3}{2}kT$, so if we assume an ideal gas we can write

$$P = \frac{NkT}{V} = \frac{2}{3}\epsilon. \tag{14}$$

Therefore, our equation from before can be rewritten to

$$3\int_0^{V_*} PdV = 2\int_0^{V_*} \epsilon dV = -\Omega. \tag{15}$$

We can write the internal energy as

$$U = \int_0^{V_*} \epsilon dV, \tag{16}$$

which finally gives the standard form of the Virial theorem:

$$U = -\frac{1}{2}\Omega \text{ or } 2U + \Omega = 0. \tag{17}$$

4. **Stellar Evolution**

(a) Sketch the Sun's evolution in the Hertzsprung-Russel (HR) diagram from the main sequence to a white dwarf. Use effective temperature and absolute luminosity for the axes, using the conventions of the HR diagram. Label the different stages.

(b) The main sequence is not a single point on the star's evolutionary track. The Sun slowly increases in luminosity during its time on the Main Sequence. Explain why.

(c) Describe the Sun's internal structure when it leaves the main sequence. What is this point called? Mark this point on your diagram.

(d) What happens at the tip of the Red Giant Branch? Explain how this process happens.

Solution:

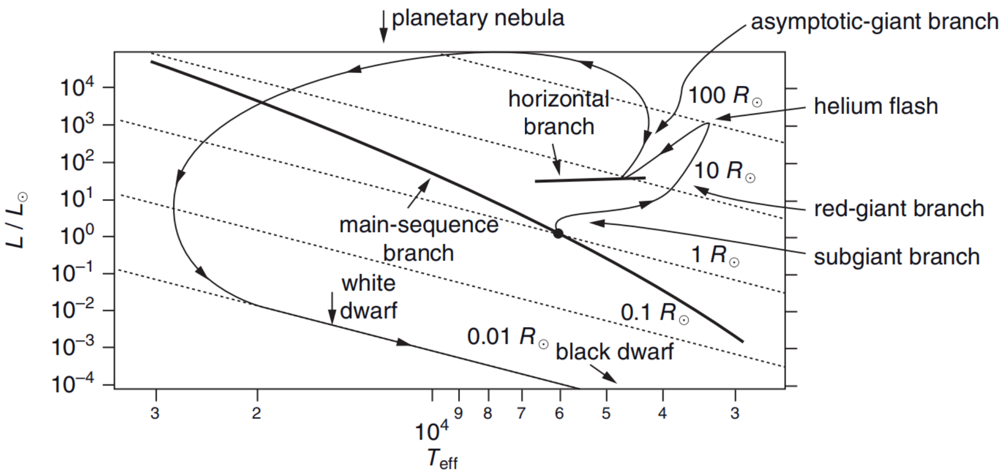
(a) 

Figure 6.13 Illustration of the approximate evolutionary track of a $1 M_\odot$ star in the H–R diagram. The dotted lines show the position for various values for the radius.

Figure 6.13 from course book by Francis LeBlanc.

(b) fusion $\rightarrow$ mean molecular weight increases $\rightarrow$ pressure decreases $\rightarrow$ contraction $\rightarrow$ gas layer pulled inwards, starting to fuse $\rightarrow$ temperature increases, Luminosity higher.

(c) The main sequence turn-off point: the core, which consist of inert He and H, has stopped fusing hydrogen $\rightarrow$ hydrogen shell burning starts $\rightarrow$ eventually He core burning starts.

(d) The helium core left behind from core hydrogen burning eventually becomes degenerate. The gas pressure in a degenerate plasma is independent of temperature which means that once the temperature increases, helium starts to fuse but it does not contribute to the outward gas pressure and therefore does not counteract the gravitational contraction. The continuing contraction causes further temperature increase, which enables more helium fusion in what could be described as a runaway effect, i.e., the "helium-flash". Eventually the temperature increase lifts the degeneracy in the core and the star stabilizes on the horizontal branch (see e.g. page 243 from the course book by Francis LeBlanc).

5. **Stellar Structure**

(a) What is the name of this equation?

$$\nabla_{\text{rad}} = \frac{3k_R}{64\pi r^2 g} \frac{P}{\sigma T^4} L(r) > \nabla_{\text{adi}} = \left(\frac{\gamma - 1}{\gamma}\right)$$

(b) Explain why this equation is important, and describe two situations where this criterion holds.

Solution:

(a) Schwarzschild Criterion.

(b) It governs whether convection occurs in a star. We see convection under two circumstances. Firstly, when the star's luminosity $L(r)$ becomes very large, and secondly, when the star's opacity k_R becomes very large. In both of these cases, the left-hand side will increase, giving a large temperature gradient, and therefore violate the Schwarzchild criterion. This will also depend on the star's internal structure, which may cause the convective movement to be damped.

6. **Radiative transfer**

(a) State the main assumption for a grey atmosphere.

(b) Using the radiative transfer equation, neglecting emissivity,

$$\frac{u}{\rho}\frac{dI_\nu(z,u)}{dz} = -k_\nu I_\nu(z,u), \tag{4}$$

Show that

$$I_\nu(z) = I_\nu^0 e^{-k_\nu \rho z}. \tag{5}$$

Solution:

(a) A grey atmosphere is an atmosphere in which the opacity is independent of frequency.

(b) Starting with the given equation, we first take $u = 1$ since we align our z-direction with the direction in which light travels. Then we can rewrite our equation to:

$$\int_{I_\nu^0}^{I_\nu(z)} \frac{dI_\nu}{I_\nu} = -\int_0^z k_\nu \rho dz. \tag{18}$$

Since k_ν and ρ are assumed to be constant, we can integrate this and get

$$\ln(I_\nu(z)) - \ln\left(I_\nu^0\right) = -k_\nu \rho z \tag{19}$$

which can be rewritten to

$$I_\nu(z) = I_\nu^0 e^{-k_\nu \rho z}. \tag{20}$$

7. **Atomic Lines**

(a) Name three different line broadening mechanisms.

(b) The Voigt-profile is the result of the convolution of two different broadening mechanisms.

   1. State which broadening mechanisms make up the Voigt-profile

2. Indicate (i.e draw a sketch) how the profiles of these broadening mechanisms result in the Voigt-profile, clearly showing where their features are most prominent.
3. The hydrogen Balmer lines are typically much broader than other lines from other atoms in a stellar spectrum. Name and explain the mechanism responsible.

(c) The equivalent width is defined as the width of a hypothetical spectral line of rectangular shape that absorbs all radiation within its width and with the same total energy associated with the line,

$$W_\lambda = \int \left[1 - \frac{F_\lambda}{F_c}\right] d\lambda = \int [1 - R_\lambda] d\lambda = \int A_\lambda d\lambda. \tag{21}$$

1. Sketch the shape of a line and how its equivalent width relates to the line.
2. Determine the equivalent width, given the following line,

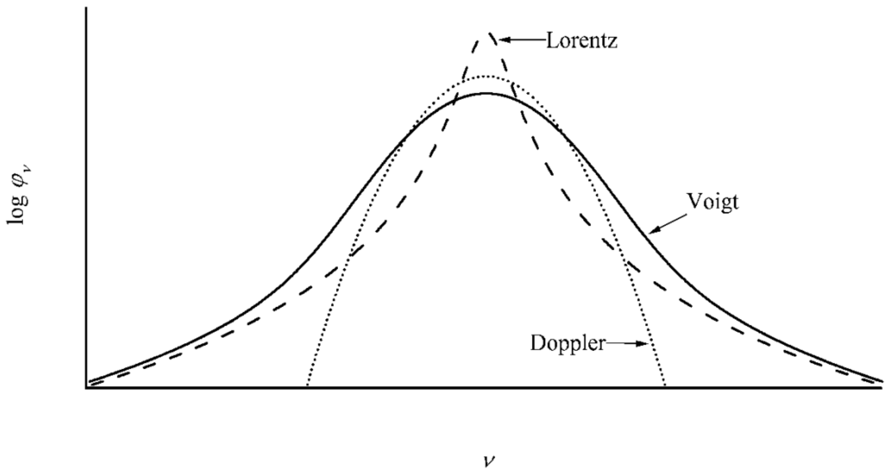
$$F_\lambda = \begin{cases} F_c \left[1 - \frac{2}{3}\cos\left(\frac{[\lambda-\lambda_0]\pi}{400\text{Å}}\right)\right], & \text{if } |\lambda - \lambda_0| \leqslant 200\text{Å}, \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

Solution:

(a) See sec 4.3 of the course book by Francis LeBlanc.

- Natural broadening: quantum uncertainty.
- Doppler broadening: Doppler shift.
- Pressure broadening: Perturbations in potential due to other particles.

(b) 1. The Voigt profile consists of a natural + Doppler profile.

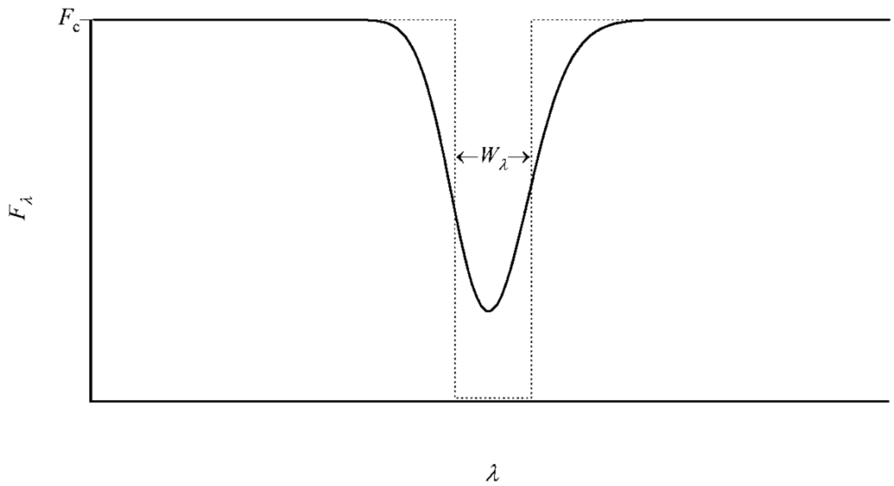


2.

3. This is because hydrogen lines are heavily influenced by the linear Stark effect (a form of pressure broadening) due to their atomic structure compared to other atoms. It is analogous to the Zeeman effect due to a magnetic field: For the linear Stark effect, due to an external electric field, splitting occurs in the degenerate energy levels. While the spectral lines caused by these split levels are too small to be resolved individually, the net effect of many split lines is observing one significantly

broadened line.
Additionally, hydrogen atoms have a low mass, implying they move faster at certain temperature, $\frac{1}{2}mv^2 = \frac{3}{2}kT$. This is sub-dominant to the Stark effect and is attributable to Doppler broadening.

(c)

1. 

2. The width of the given profile can be calculated as follows:

$$W_\lambda = \int_{\lambda_0 - 200\text{Å}}^{\lambda_0 + 200\text{Å}} \left[1 - \left(1 - \frac{2}{3}\cos\left(\frac{\pi(\lambda - \lambda_0)}{400\text{Å}}\right)\right)\right] d\lambda \tag{22}$$

$$= \frac{2}{3}\int_{\lambda_0 - 200\text{Å}}^{\lambda_0 + 200\text{Å}} \cos\left(\frac{\pi(\lambda - \lambda_0)}{400\text{Å}}\right) d\lambda \tag{23}$$

$$= \frac{800\text{Å}}{3\pi} \sin\left(\frac{\pi(\lambda - \lambda_0)}{400\text{Å}}\right)\Bigg|_{\lambda_0 - 200\text{Å}}^{\lambda_0 + 200\text{Å}} \tag{24}$$

$$\approx 170\text{Å}. \tag{25}$$

Note that the continuum F_c is set to 1 such that $W_\lambda = \int (F_c - F_{\text{line}})\mathrm{d}\lambda$.

8. **Polytropic Models**

One of the well-known equations in stellar structure theory is the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{d\theta(\xi)}{d\xi}\right) = -\theta^n(\xi), \tag{8}$$

where,

$$\xi = r\sqrt{\frac{4\pi G\rho_c^2}{(n+1)P_c}}, \tag{9}$$

a dimensionless quanity.

(a) The solution to this equation for n=0 is $\theta(\xi) = 1 - \frac{\xi^2}{6}$. Find ξ_0 such that $\theta(\xi_0) = 0$. At what location is ξ_0 defined for a star?

(b) A good approximation ($r \leq R_\odot/2$) of the inner workings of a star is given by a polytropic star with $n = 5$. Eq. (8) then has the solution:

$$\theta(\xi) = \frac{1}{\left(1 + \frac{\xi^2}{3}\right)^{1/2}}.$$

Can you reason why the approximation fails to describe a real star at $r > R_\odot/2$?

(c) Derive the stellar mass $M(r)$ for a polytropic star with $n = 5$, write your answer in terms of ρ_c, r and ξ. (Hint: A polytropic density profile is written as $\rho(r) = \rho_c\theta^n(r)$)

Solution: See section 5.4.

(a) The equation above has $\theta = 0$ for $\xi_0 = \pm\sqrt{6}$. This is of course positive since $r \geq 0$, so $\xi_0 = \sqrt{6}$ and corresponds to the surface of the polytropic star. This gives us a relation for R_* from Eq. (9):

$$R_* = \alpha\xi_0$$

where $\alpha = \sqrt{\frac{(n+1)P_c}{4\pi G\rho_c^2}}$.

(b) By looking at the solution given, there is no ξ_0 for which $\theta = 0$. This implies a ploytropic star with infinite radius (the radius being defined as the distance at which the density is zero), evidently this approximation starts to break down when comparing to extensive stellar models with a finite radius $R_\odot$.

(c) The stellar mass of any spherical star can be calculated through

$$M(r) = 4\pi \int_0^r r'^2 \rho(r')dr'.$$

We'll have to apply a change of variables $r \to \xi$. The radius r corresponds to ξ and $r^2 = (\xi\alpha)^2$. Now $\rho(r)dr = \rho(\xi)d\xi \to \rho(\xi) = \rho(r)\left|\frac{\mathrm{d}r}{\mathrm{d}\xi}\right| = \rho_c\theta^n(\xi) \cdot \alpha$. Setting $n = 5$, the following integral arises

$$\begin{aligned} M(r) &= 4\pi\rho_c\alpha \int_0^\xi r^2\theta^5(\xi')d\xi' \\ &= 4\pi\rho_c\alpha \int_0^\xi (\xi'\alpha)^2 \left(1 + \frac{\xi'^2}{3}\right)^{-5/2} d\xi' \\ &= 4\pi\rho_c\alpha^3 \int_0^\xi \frac{\xi'^2}{\left(1 + \frac{\xi'^2}{3}\right)^{5/2}} d\xi'. \end{aligned}$$

Use the handy integral:

$$\begin{aligned} M(r) &= 4\pi\rho_c\alpha^3 \left[\frac{\sqrt{3}\xi^3}{(\xi^2+3)^{3/2}}\right] \\ &= 4\pi\rho_c r^3 \left(\frac{\sqrt{3}}{(\xi^2+3)^{3/2}}\right). \end{aligned}$$

Handy integral

$$\int \frac{x^2}{\left(\frac{x^2}{3}+1\right)^{5/2}}dx = \frac{\sqrt{3}x^3}{(x^2+3)^{3/2}} + C$$