Practice session: Astrophysical Hydrodynamics

Kapteyn Learning Community Sirius A
Give motivations and/or derivations for your answers.
Note: Some problems are adapted from past Homeworks or Filippo Fraternali's lecture notes and quiz-type questions.

Part 1: Multiple choice questions

1. Consider a small (non self-gravitating) cold gas cloud $T < 10^4$ K embedded into a vey hot $T > 10^6$ K medium. Which hydrodynamical effect do you expect to dominate the evolution of the cloud?
   (a) Radiative cooling
   (b) Thermal conduction
   (c) Convection
   (d) Hydrostatic equilibrium

2. In a dwarf galaxy, a fraction of interstellar medium is heated at high temperature ($T \approx 10^6$ K) by supenova explosions. Under what condition can we guess that this hot gas can escape the galaxy?
   (a) If it can radiate very efficiently
   (b) If it evolves adiabatically
   (c) If its sound speed is larger than the escape speed from the galaxy
   (d) If its thermal speed is of the order of its sound speed

3. Consider a region of the interstellar medium of a galaxy where star formation is taking place. If we want to describe the gas content using the continuity equation, we need:
   (a) to consider its Lagrangian form
   (b) to add source terms on the right hand side
   (c) to use the equation of state of ideal gases
   (d) to add sink terms on the right hand side

4. The internal energy of a parcel of gas can increase as a consequence of which of the following phenomena? (more than one answer can be correct):
   (a) Advection of the fluid
   (b) Radiative cooling
   (c) Being subject to compression work
   (d) Thermal conduction

5. Consider the accretion disc around a black hole and assume that the black hole fully dominates the gravitational potential. Which of the following statements is correct?
   (a) The disc cannot be in hydrostatic equilibrium
   (b) If the disc is in hydrostatic equilibrium, its thickness is likely constant with radius
   (c) If the disc is in hydrostatic equilibrium, its thickness likely increases with radius
   (d) The disc is self-gravitating

6. As a consequence of the passage of a strong shock wave (shock speed $v_s \approx 1000$ km/s), the interstellar medium of a galaxy is perturbed. Which of the following statements is correct?
   (a) Immediately behind the shock the density grows by orders of magnitude

(b) Immediately behind the shock the temperature grows by orders of magnitude
(c) If the perturbed medium radiates efficiently, the density grows by a factor 4
(d) In adiabatic shocks, the entropy is conserved

7. Consider a shock wave moving at a speed $v_s \approx 500$ km/s through the atomic neutral medium of the Milky Way. Can such a shock ionise the gas?
   (a) Yes, but only if the gas radiates efficiently
   (b) No, but it can produce excitation
   (c) Yes, and that would be called collisionally ionised gas
   (d) No in general, but yes if the gas has a low density

8. At the end of the evolution of a supernova remnant, can some kinetic energy be transferred to the ISM to feed the gas turbulence?
   (a) Yes, nearly all the initial energy of 10^{51} erg, as a consequence of momentum conservation
   (b) Yes, but only about half of the initial energy due to energy equipartition
   (c) Yes, but only a small percentage of the initial energy, due to radiative losses
   (d) No, all the initial energy is converted into heat

9. According to the Sedov solution of a blast wave, the shock propagates through the medium in which of the following ways?
   (a) It accelerates pushed by the high pressure in the internal bubble
   (b) It decelerates as it perturbs more and more initially unperturbed gas
   (c) It expands at nearly constant speed, which is about 3/4 of the initial shock speed
   (d) The regimes of acceleration or deceleration depend on the cooling time in the medium right behind the shock wave

10. Consider a supernova remnant produced by a standard supernova explosion (normal energy input), but taking place in an underdense region of the ISM. Which of the following statements is correct? (More than one answer can be correct)
   (a) The cooling time of the perturbed medium is shorter and the Sedov phase lasts longer
   (b) The cooling time of the perturbed medium is larger and the Sedov phase last longer
   (c) The radius grows in time during the radiative phase with the same dependence (slope) it would have in the normal ISM
   (d) The supernova remnant radiates away all its initial energy

11. Consider a cold cloud in the interstellar medium nearly at constant temperature and having a mass two orders of magnitude smaller than the Jeans mass, calculated given its temperature and density. What can we conclude about its dynamical state?
   (a) The cloud is gravitationally unstable and it should collapse in a free-fall time
   (b) The cloud is gravitationally unstable, but it can be stabilised by turbulence or magnetic fields
   (c) The cloud is not gravitationally unstable, but it can become convectively unstable
   (d) The cloud must be pressure confined

12. Consider a portion of warm, $T \approx 10^4$ K, ISM in the path of a hot, $T \approx 10^6$ K, wind produced by massive stars in its vicinity. The wind is moving at fast speeds: hundreds of km/s. What kind of instability will likely develop at the surface of the warm gas?
   (a) Rayleigh-Taylor instability
   (b) Kelvin-Helmholtz instability

(c) Thermal instability
(d) Difficult to say without calculating the Jeans mass of the hot gas

13. The region of space around galaxies is permeated by rarefied gas at a few million degrees, but some observations have also shown the presence of clouds of relatively cold $T \approx 10^4$ K gas. These clouds have masses much smaller than the Jeans mass at their temperature and density. Their lifetime is estimated to be quite long, at least a few hundred Myr. What are the effects that can help them survive? (More than one answer can be correct)
    (a) Self-gravity
    (b) Radiative cooling
    (c) Pressure confinement from the external medium
    (d) Convection

14. Which of the following effects can occur as a consequence of thermal instability?
    (a) A portion of gas in the hot halo of a galaxy initially in hydrostatic equilibrium can cool radiatively and fall towards the centre of the potential well
    (b) Gas at intermediate (unstable) temperatures can be found in interface regions, between hot and cold gas
    (c) The interior of a star can become unstable and release large amount of radiation
    (d) The various phases of the ISM can reach pressure equilibrium

15. The presence of viscosity in a fluid introduces important differences with respect to an ideal (inviscid) fluid. Which of the following statements are correct? (More than one answer can be correct)
    (a) In the presence of viscosity some kinetic energy is turned into internal energy
    (b) If shear viscosity is present, Kelvin-Helmholtz instability will develop more rapidly
    (c) Bulk viscosity is only important in incompressible fluids
    (d) The Euler equation is no longer valid and we need to add some viscous terms

16. Which of these statements about the Kolmogorov theory of turbulence and its applications is correct? (More than one answer can be correct)
    (a) Turbulent entities, eddies, are fully developed at all possible scales down to the dissipation scale
    (b) Energy must be continuously injected in the small scales and transferred to the largest ones
    (c) If the medium is compressible, the energy dissipation will be always more efficient than Kolmogorov predicts
    (d) In the largest eddies, viscosity must be important

17. Consider a molecular cloud with the presence of an internal magnetic field. What effect can this magnetic field have on the dynamics of the cloud?
    (a) It can enhance the radiative cooling
    (b) It can counteract gravity through magnetic pressure
    (c) It can facilitate the collapse through emission of Alfvén waves
    (d) A magnetic field cannot be important in a molecular clouds because there are no free electrons

Solution:

1. (b): too cold for radiative cooling, hydrostatic equilibrium is a state not a process, and convection requires large-scale buoyancy.

2. (c): allows it to escape the gravitational influence.
3. (d): stars are taking away mass (if there was strong stellar wind activity, then you would add a source term).
4. (a), (c), (d): radiative cooling decreases the internal energy.
5. (c): flaring effect, $dP/dz = -\rho\Omega z \Rightarrow h \propto c_s/\Omega \Rightarrow \Omega \propto R^{-3/2} \Rightarrow h \propto R^{3/2}$
6. (b), (d): $T \propto v_s^2$, adiabatic gives $\rho_1 = 4\rho_0$ and entropy increases.
7. (c): $T \approx 10^5\,\mathrm{K}\,(v_s/100\,\mathrm{km\,s^{-1}})^2 \approx 2 \times 10^6$ which is greater than $T_{\text{ionisation}} = 13.6\,eV/K = 10^5$ K.
8. (c): about 3-10%
9. (b): $r_s \propto t^{2/5} \Rightarrow v_s \propto t^{-3/5}$
10. (b): $t_{\text{cool}} \propto n^{-1}$, so cooling time increases for lower density (slope also remains the same).
11. (d): since $M < M_J$, gravity is not important (so no convection either).
12. (b): shear is the main driver (gravity not important so no RT or Jeans, thermal not the primary driver).
13. (b), (c): since $M < M_J$, gravity and convection are not important.
14. (a): slight cooling leads to sudden pressure loss, which leads to even more cooling.
15. (a), (d): friction (viscosity smooths out velocity gradients so no KH) + Euler eq. needs to be replaced with the Navier-Stokes eq.
16. (a), (c): for large eddies, viscosity becomes less important, cascade
17. (b): Alfven waves can provide more support by opposing compression.

Part 2: Problems

1. **What is a fluid anyway?**
Hydrodynamics deals with the physics of fluids, but what is a fluid? And can we always use these concepts?
   (a) Describe what a fluid is in the context of Astrophysical Hydrodynamics.
   (b) Discuss and compare the relevant length scales.
   (c) Explain why the fluid approximation is good for the solar interior, but will not be a good description for the solar wind. Consider that the mean free path inside the Sun is of the order of 10^{-4} cm and can be greater than 1 AU for the solar wind.

Solution:

(a) The fluid is an idealized concept in which the matter is described as a continuous medium with certain macroscopic properties that vary as continuous function of position (e.g., density, pressure, velocity, entropy).

(b) One assumes that the scales L over which these quantities are defined is much larger than the size of a fluid element λ, which in turn is much larger than the mean free path of the individual particles that constitute the fluid, l:

$$L \gg \lambda \gg l;\ \lambda = \frac{1}{\sigma n}, \tag{1}$$

where n is the number density of particles in the fluid and σ is a typical interaction cross section.

(c) For the solar interior, this mean free path is on the order of 10^{-4} cm, which is much smaller than the scale of the solar interior, so the fluid approximation holds here.

For the solar wind the mean free path will be much larger than an AU, so here it will not hold.

2. **Lagrangian and Eulerian view on life**
What is the Eulerian description of fluid dynamics, and what is the Lagrangian description. Describe the physical meaning and difference between Eulerian derivative of a thermodynamic quantity Q,

$$\frac{\partial Q}{\partial t} \tag{2}$$

and the Lagrangian derivative

$$\frac{DQ}{Dt} \tag{3}$$

and state the relationship between these two derivatives for a fluid with flow field $\vec{v}$.

Solution: The Eulerian description of fluid dynamics follows the properties of the fluid at fixed points in space, such that the fluid is free to move along these points.

The Lagrangian description of fluid dynamics follows the properties of the fluid on points fixed on the fluid, such that the properties can always be assumed to be associated with the same fluid element.

$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \vec{v} \cdot \nabla Q \tag{4}$$

3. **The symbols... what do they mean?**
Consider the following equation of hydrodynamics:

$$\rho T \frac{D\mathcal{S}}{Dt} = -\boldsymbol{\nabla} \cdot \mathbf{q} - \rho \mathcal{L} \tag{5}$$

(a) Describe what it represents and what is the quantity $\mathcal{S}$.

(b) What do the two terms on the right hand side represent?

(c) How can the $\mathcal{L}$ term be further developed?

Solution:

(a) This is the heat equation, one of the many forms of the energy equation. The quantity $\mathcal{S}$ is the specific entropy of the fluid, or the entropy per unit volume.

(b) $-\nabla \cdot \mathbf{q}$ is the specific change in energy due to conduction
$\rho\mathcal{L}$ is the specific change in energy due to radiation

(c) $\mathcal{L} = \Lambda - \Gamma$, where Λ is the cooling function and Γ is the heating function.

4. **Star goes boom**
When a massive star reaches the end of its life, it will violently blast a large part of itself into outer space. Indeed, these are one the most violent events in the universe. The evolution of the supernova remnant, as it propagates through the ISM, can be described in terms of three phases.

(a) Which are these three phases, and what is the time evolution of the supernova remnant shell radius r for the first two phases?

(b) Discuss the physics of the different stages, and the transitions between them.

Solution: Free Phase: This is the first phase of a supernova explosion, and describes how the supernova shockwave expands in the regime where the energy released by the supernova is far greater than the energy lost due to interactions with the surrounding ISM: $r \propto t$.

Sedov (adiabatic) Phase: In this stage the shockwave (SW) has accelerated the ISM surrounding the supernova. The SW now expands adiabatically, i.e. energy losses due to radiation can still be ignored. The energy losses of the SW become significant. Consequently, the SW is decelerated considerably. The scale of the bubble increases over time as $r \propto t^{2/5}$.

Radiation Phase: As the bubble expands and the outer regions cool off, the effects due to radiation can no longer be ignored. The SW goes from *adiabatic* expansion to *isothermal* expansion. In fact, radiation will become more and more important over time. The density inside the bubble roughly increases of the order of the Mach Number squared and its velocity almost equals the SW velocity. If we ignore the pressure of the inner bubble then we can describe the SW evolution through the *snowplough* effect. The SW then propagates through space while accumulating mass and subsequently slowing down continuously. Eventually, the supernova remnant slowly starts to merge with the ISM, and practically *dissipates* away.

5. **More Supernova explosions!**
In this exercise we are going to have a deeper look at the physics of supernova remnant Shock Waves and what effect it has on the surrounding medium.

(a) Write down the Rankine-Hugoniot jump equations

(b) Explain the origin of each of the equations

(c) The location of the supernova implies that it happens in an isothermal environment. Derive the compression factor between the initial and final medium and find the solution for the velocities. (Tip: use that the pressure becomes $P(\rho) = \rho c_s^2$, with c_s the speed of sound)

Solution:

(a) They are:

$$\rho_0 u_0 = \rho_1 u_1 \tag{6}$$
$$\rho_0 u_0^2 + P_0 = \rho_1 u_1^2 + P_1 \tag{7}$$
$$\frac{u_0^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0} = \frac{u_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1}, \tag{8}$$

where $P = k_B \rho^\gamma$

(b) These are derived from the conservation equations, as those equations still hold during a shock. There are derived from the conservation of mass, the conservation of momentum and the conservation of energy respectively.

(c) We know the temperature will stay constant, so we only need the first two shock conditions which (by dividing both sides by c_s and c_s^2 for the first and second condition respectively) turn into:

$$\frac{\rho_0}{\rho_1} = \frac{\mathcal{M}_1}{\mathcal{M}_0}, \tag{9}$$
$$\rho_0 \mathcal{M}_0^2 + \frac{P_0}{c_s^2} = \rho_1 \mathcal{M}_1^2 + \frac{P_1}{c_s^2}, \tag{10}$$

where M now denotes the mach number $\mathcal{M} := u/c_s$. If we now fill in the pressure using the hint we get:

$$\rho_0 \mathcal{M}_0^2 + \rho_0 = \rho_1 \mathcal{M}_1^2 + \rho_1$$

Now we can combine the two equations to get as a relation for the two velocities:

$$\mathcal{M}_0\left(\mathcal{M}_1^2 + 1\right) = \mathcal{M}_1\left(\mathcal{M}_0^2 + 1\right)$$

Solving this leads to either $\mathcal{M}_0 = \mathcal{M}_1$ (no shock so useless) or $\mathcal{M}_0 = \frac{1}{\mathcal{M}_1}$ (the solution we want). This also implies that the compression factor becomes: $\frac{\rho_1}{\rho_0} = \mathcal{M}_0^2$

6. **Space tornadoes and Kelvin's circulation theorem**

In the midst of chaos you find yourself pondering about a potential danger. No one in your spaceship remotely has the intellectual capacity to even fathom such a crisis. As the ship drifts through turbulent and dense ISM, and your colleagues fall asleep, you find yourself determined to recover the tool that may give insight to possible impending doom. *Space tornadoes.*

(a) For a fluid with flow velocity field $\vec{\mathbf{u}}$, write the Eulerian description of the Euler equation, taking into account the influence of the gravity field $\vec{\mathbf{g}}$ and the gas pressure P. Ignore viscous effects.

(b) Show that for a *barotropic* flow in a conservative gravitational field (i.e. $\vec{g} = \vec{\nabla}\phi$) we have:

$$\frac{\partial \vec{\omega}}{\partial t} + \boldsymbol{\nabla} \times (\vec{\omega} \times \vec{\mathbf{u}}) = 0 \tag{11}$$

where $\vec{\omega}$ is the *vorticity*.
Hint: consider the identities

$$\boldsymbol{\nabla}\frac{1}{2}a^2 = (\vec{\mathbf{a}} \cdot \boldsymbol{\nabla})\,\vec{\mathbf{a}} + \vec{\mathbf{a}} \times (\boldsymbol{\nabla} \times \vec{\mathbf{a}}) \tag{12}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \vec{\mathbf{a}}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \vec{\mathbf{a}}) - \nabla^2 \vec{\mathbf{a}} \tag{13}$$

(c) You define the flow's *circulation* Γ around a circuit C (with surface area A) by

$$\Gamma := \oint_C \vec{\mathbf{u}} \cdot d\vec{\mathbf{l}} = \int_A \vec{\omega} \cdot d\vec{\mathbf{A}}, \tag{14}$$

how does Γ change with time?
Use that:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \int_A \frac{\partial \vec{\omega}}{\partial t} \cdot d\vec{\mathbf{A}} + \oint (\vec{\omega} \times \vec{\mathbf{u}}) \cdot d\vec{\mathbf{l}} \tag{15}$$

The resulting relation is *Kelvin's circulation theorem*. What does it imply for inviscid barotropic space tornadoes and should you and your crew be scared?

Solution:

(a) This is given by

$$\frac{\partial \vec{\mathbf{u}}}{\partial t} + \left(\vec{\mathbf{u}} \cdot \vec{\nabla}\right) \vec{\mathbf{u}} = -\frac{\boldsymbol{\nabla} P}{\rho} + \vec{\mathbf{g}} \tag{16}$$

Where the viscous effects have been ignored. In fact, this would've led to a Navier-Stokes eq. without a potential gravity field.

(b) In our case, we start by taking the curl of both sides of the equation mentioned above (also substituting the gravity thing):

$$\boldsymbol{\nabla} \times \frac{\partial \vec{\mathbf{u}}}{\partial t} + \boldsymbol{\nabla} \times \left(\vec{\mathbf{u}} \cdot \vec{\nabla}\right) \vec{\mathbf{u}} = -\boldsymbol{\nabla} \times \frac{\boldsymbol{\nabla} P}{\rho} + \boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi \tag{17}$$

Note that for the right hand side $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 0$ for any ϕ, and by applying the identity,

$$-\boldsymbol{\nabla} \times \frac{\boldsymbol{\nabla} P}{\rho} = -\boldsymbol{\nabla}\frac{1}{\rho} \times \boldsymbol{\nabla} P \tag{18}$$

$$= -\frac{1}{\rho^2}\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P \tag{19}$$

$$= 0 \tag{20}$$

since $\boldsymbol{\nabla}\rho = \boldsymbol{\nabla} P$ for barotropic fluids, which makes the cross product zero. This leaves us with

$$\boldsymbol{\nabla} \times \frac{\partial \vec{\mathbf{u}}}{\partial t} + \boldsymbol{\nabla} \times \left(\vec{\mathbf{u}} \cdot \vec{\nabla}\right) \vec{\mathbf{u}} = 0 \tag{21}$$

Recalling the definition of vorticity ($\omega = \boldsymbol{\nabla} \times \vec{\mathbf{u}}$) and using the identity given in Equation 12:

$$\frac{\partial \vec{\omega}}{\partial t} + \boldsymbol{\nabla} \times \left[\boldsymbol{\nabla}\frac{1}{2}u^2 - \vec{\mathbf{u}} \times \underbrace{(\boldsymbol{\nabla} \times \vec{\mathbf{u}})}_{=\vec{\omega}}\right] = 0 \tag{22}$$

$$\frac{\partial \vec{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\vec{\mathbf{u}} \times \vec{\omega}) = 0 \tag{23}$$

$$\frac{\partial \vec{\omega}}{\partial t} + \boldsymbol{\nabla} \times (\vec{\omega} \times \vec{\mathbf{u}}) = 0 \tag{24}$$

(c) Apply Stokes' thm. ($\oint \vec{\mathbf{A}} \cdot d\vec{\mathbf{l}} = \int_A \left(\boldsymbol{\nabla} \times \vec{\mathbf{A}}\right) \cdot d\vec{\mathbf{A}}$) to find that

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \int_A \frac{\partial \omega}{\partial t} \cdot d\vec{\mathbf{A}} + \int_A \boldsymbol{\nabla} \times (\vec{\omega} \times \vec{\mathbf{u}}) \cdot d\vec{\mathbf{A}} \tag{25}$$

$$= \int_A \left[\frac{\partial \vec{\omega}}{\partial t} + \boldsymbol{\nabla} \times (\vec{\omega} \times \vec{\mathbf{u}})\right] \cdot d\vec{\mathbf{A}} \tag{26}$$

$$= 0 \tag{27}$$

Thus, in inviscid barotropic fluids, vorticity is conserved for all time. Vorticity is never created and they never dissipate. Only when the surroundings start to cause viscous effects and/or the pressure seems to change does $\frac{\mathrm{d}\Gamma}{\mathrm{d}t} \neq 0$. If you do happen to encounter an inviscid barotropic space tornado ... you better book it boi.

7. **Atmospheres of galaxy clusters**

Galaxy clusters are the largest and most massive structures in the Universe. Their main mass component is dark matter but they also have hot gas, while the stellar component in galaxies can be considered negligible. Here, we assume that the gravitational potential is fully dominated by the dark matter and can be written as

$$\Phi(r) = v_c^2 \ln\left(\frac{r}{r_0}\right) + \Phi_0, \tag{28}$$

where v_c is a parameter (the circular speed), r_0 is a reference radius and Φ_0 is the value of the potential at that radius.

(a) Assume that the hot gas has an isothermal equation of state ($P = c_s^2\rho$, with c_s constant) and find its radial density profile. How does this profile change if we change the gas temperature?

(b) Assuming that the gas temperature is $T = 10^8$ K, derive the cooling time as a function of r. Estimate (approximately) the value of the *cooling radius* (distance from the centre of the cluster at which the cooling time is equal to the Hubble time). For this calculation you can use $r_0 = 100$ kpc, $n_{\mathrm{e},0} = 10^{-3}\,\mathrm{cm}^{-3}$ (electron density at r_0) and $v_s = \sqrt{2}c_s$. The cooling function of collisional ionisation equilibrium at $T = 10^8$ K is $\Lambda(10^8\,\mathrm{K}) = 2.5 \times 10^{-23}\,\mathrm{erg\ cm^3\ s^{-1}}$.

(c) Assume then that the gas has, instead, a polytropic equation of state: $P = A\rho^\beta$ (with A and β constant parameters, $\beta > 1$), and determine the density profile in this case. Is the temperature constant or a function of r in this case?

Solution:

(a) Hydrostatic equilibrium gives:

$$\frac{dP}{dr} = -\rho\frac{d\Phi}{dr}. \tag{29}$$

For an isothermal gas $P = c_s^2\rho$:

$$c_s^2\frac{d\rho}{dr} = -\rho\frac{d\Phi}{dr}. \tag{30}$$

Taking the derivative with respect to r for the given expression of the potential gives:

$$\frac{d\Phi}{dr} = \frac{v_c^2}{r}, \tag{31}$$

Using the above:

$$\frac{1}{\rho}\frac{d\rho}{dr} = -\frac{v_c^2}{c_s^2}\frac{1}{r}. \tag{32}$$

$$\rho(r) = \rho_0 \left(\frac{r}{r_0}\right)^{-v_c^2/c_s^2}. \quad (33)$$

Since $c_s^2 = k_B T/\mu m_p$,

$$\rho(r) \propto r^{-\frac{v_c^2}{c_s^2}} \propto r^{-\frac{v_c^2}{T}}. \quad (34)$$

Higher $T \Rightarrow$ shallower profile, and lower $T \Rightarrow$ steeper profile.

(b) The cooling time is:

$$t_{\mathrm{cool}} = \frac{3}{2}\frac{n k_B T}{n_e n_i \Lambda}. \quad (35)$$

Assuming $n_e \approx n_i \Rightarrow n \approx 2n_e \propto \rho$, we get:

$$t_{\mathrm{cool}} \propto \frac{T}{n\Lambda}. \quad (36)$$

From part (a), $n(r) \propto r^{-\frac{v_c^2}{c_s^2}}$, hence:

$$t_{\mathrm{cool}}(r) \propto r^{\frac{v_c^2}{c_s^2}}. \quad (37)$$

Using $v_c^2 = 2c_s^2$:

$$t_{\mathrm{cool}}(r) \propto r^2. \quad (38)$$

At r_0:

$$t_{\mathrm{cool},0} \approx \frac{3k_B T}{n_{e,0}\Lambda}. \quad (39)$$

Plugging in values:

$$t_{\mathrm{cool},0} \approx \frac{3 \times (1.38 \times 10^{-16})(10^8)}{(10^{-3})(2.5 \times 10^{-23})} \approx 10^{18}\ \mathrm{s} \approx 6 \times 10^{10}\ \mathrm{yr}. \quad (40)$$

Thus:

$$t_{\mathrm{cool}}(r) = t_{\mathrm{cool},0}\left(\frac{r}{r_0}\right)^2 \Rightarrow r = r_0\sqrt{\frac{t_{\mathrm{cool}}(r)}{t_{\mathrm{cool},0}}} \quad (41)$$

Setting $t_{\mathrm{cool}} = t_H = 1.38 \times 10^{10}$ yr:

$$r \approx 50\ \mathrm{kpc}. \quad (42)$$

(c) For a polytropic equation of state, hydrostatic equilibrium gives:

$$A\beta\rho^{\beta-1}\frac{d\rho}{dr} = -\rho\frac{v_c^2}{r}. \quad (43)$$

Rewriting:

$$\frac{d\rho}{dr} = -\frac{v_c^2}{A\beta}\frac{1}{r}\rho^{2-\beta}. \quad (44)$$

Separating variables and integrating:

$$\rho^{\beta-2}d\rho = -\frac{v_c^2}{A\beta}\frac{dr}{r}. \quad (45)$$

$$\frac{\rho^{\beta-1} - \rho_0^{\beta-1}}{\beta-1} = -\frac{v_c^2}{A\beta}\ln\left(\frac{r}{r_0}\right) \quad (46)$$

$$\rho(r) = \rho_0 \left[1 - \frac{\beta - 1}{\beta} \frac{v_c^2}{A\rho_0^{\beta-1}} ln\left(\frac{r}{r_0}\right) \right]^{\frac{1}{\beta-1}} \tag{47}$$

Since $k_B T/\mu m_p = P/\rho$, the temperature is given as:

$$T = \frac{\mu m_p}{k_B} A\rho^{\beta-1}, \tag{48}$$

which varies with r.

8. **Navier and Stokes in a secret tunnel**

We are going to investigate the Navier-Stokes equation, and its application towards the flow through a cylindrical pipe. The Navier-Stokes equation is an extension, and a more complete, version of the Euler equation now also including an internal friction term. the Navier-Stokes equation is given by:

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \eta\nabla^2\mathbf{u}. \tag{49}$$

One situation in which we can derive an analytical expression for a viscous flow is that of a steady, laminar flow through a pipe (or secret tunnel, through the mountain) of radius R. Steady means that the time derivative of the relevant physical quantities is zero $\partial/\partial t = 0$. For the pipe we use cylindrical coordinates, x along the length of the pipe, radius r and sectional angle θ.

The flow is only along the length of the pipe, $u_x = u$. The radial and angular components of the fluid velocity are zero: $u_r = u_\theta = 0$. The flow is axisymmetric, and can only vary in the radial direction or, possibly, along the x-direction, that is $\partial/\partial\theta = 0$. The flow is fully developed and will not vary along the x-direction as the diameter of the pipe is constant along its length: $\partial u/\partial x = 0$

(a) Show that the Navier-Stokes equation for this situation can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\eta}\frac{\partial p}{\partial x}, \tag{50}$$

$$0 = \frac{1}{\eta}\frac{\partial p}{\partial r}. \tag{51}$$

(b) Argue that the pressure drop along the x-direction in the pipe is linear, i.e. that $(\partial p/\partial x) =$ constant, and thus,

$$\frac{\partial p}{\partial x} = -\frac{\Delta p}{L}. \tag{52}$$

Where Δp is the pressure drop along a length L of the pipe.

(c) Subsequently, show that the general solution for this equation can be written as,

$$u = -\frac{1}{4\eta}\frac{\Delta p}{L}r^2 + c_1 \ln r + c_2. \tag{53}$$

Where c_1 and c_2 are integration constants.

(d) From the boundary condition that the velocity v is finite at $r = 0$, it follows that $c_1 = 0$. Show that from the no slip boundary condition at the wall of the pipe, i.e. $u_x = u = 0$ at $r = R$, that,

$$c_2 = \frac{1}{4\eta}\frac{\Delta p}{L}R^2. \tag{54}$$

(e) And that therefore the generic solution for the flow field is given by,

$$u(r) = \frac{1}{4\eta}\frac{\Delta p}{L}(R^2 - r^2). \tag{55}$$

This is called the Hagen-Poiseuille equation. Describe and explain how the flow field $u(r)$ in the pipe behaves as function of radius r.

(f) Show that for a fluid with a density ρ the amount of mass transported through a surface of constant x per unit time is given by

$$\dot{M} = \int_0^R 2\pi\rho u r \mathrm{d}r = \frac{\pi\rho\Delta p}{8\eta L}R^4. \tag{56}$$

Solution:

(a) Both terms on the left hand side of the N-S equation vanish: the flow is fully developed, and thus $\frac{\partial \mathbf{u}}{\partial t}$ is zero. Furthermore, all derivatives in $(\mathbf{u}\cdot\nabla)\mathbf{u}$ are zero.

Since $\frac{\partial}{\partial\theta}$ is zero for all quantities, the first term on the right hand side is equal to $-\nabla p = -(\frac{\partial p}{\partial x}\hat{x} + \frac{\partial p}{\partial r}\hat{r})$

The last term is a bit more complicated, as we are working in cylindrical coordinates. However, it turns out that all terms but one vanish, and we are left with:

$$\nabla^2\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\hat{x}. \tag{57}$$

resulting in,

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \eta\nabla^2\mathbf{u}. \tag{58}$$

$$0 + 0 = -\boldsymbol{\nabla} p + \eta\boldsymbol{\nabla}^2\mathbf{u} \tag{59}$$

$$0 = -\left(\frac{\partial p}{\partial x}\hat{x} + \frac{\partial p}{\partial r}\hat{r}\right) + \eta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\hat{x}. \tag{60}$$

Therefore, if we split this up into separate equations for both coordinates, we get:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\hat{x} = \frac{1}{\eta}\frac{\partial p}{\partial x}\hat{x}, \tag{61}$$

$$0 = \frac{1}{\eta}\frac{\partial p}{\partial r}\hat{r}. \tag{62}$$

(b) If we look at equation 61, we see that the left hand side is only dependant on the radius (since u is only dependant on r). The right hand side, however, is only dependant on x (see Equation 62, and therefore both sides must be the same constant value. Thus, the change in pressure over x is a constant and the pressure is linearly dependant on x.

(c) For this we will integrate equation 61 twice:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{1}{\eta}\frac{\Delta p}{L}, \tag{63}$$

$$r\frac{\partial u}{\partial r} = -\frac{\Delta p}{2\eta L}r^2 + c_1, \tag{64}$$

$$u(r) = -\frac{\Delta p}{4\eta L}r^2 + c_1 \ln r + c_2 \tag{65}$$

$$\tag{66}$$

(d) If we fill in the given condition, we get:

$$0 = -\frac{\Delta p}{4\eta L}R^2 + 0 + c_2, \tag{67}$$

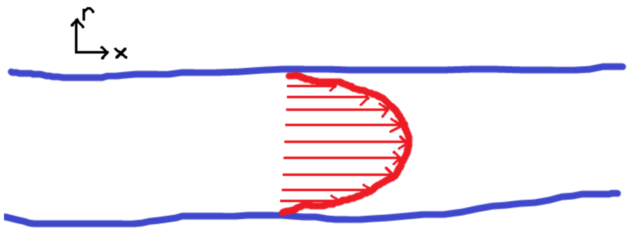
$$c_2 = \frac{\Delta p}{4\eta L}R^2 \tag{68}$$

(e)

$$u(r) = -\frac{\Delta p}{4\eta L}r^2 + c_1 \ln r + \frac{\Delta p}{4\eta L}R^2 \tag{69}$$

$$= \frac{1}{4\eta}\frac{\Delta p}{L}(R^2 - r^2). \tag{70}$$

The fluid will flow the fastest in the middle of the pipe, and then parabolically decline toward the outer edges, such that it is zero at $r = R$. See the illustration below.



Ceci n'est pas une pipe

(f) The amount of mass per unit surface transported through a point at a certain radius is given by $\rho u(r)$. To find the amount of mass flowing through a surface in the pipe, we simply integrate over the surface S, perpendicular to the x axis

$$\dot{M} = \int_S (\rho\vec{\mathbf{u}}) \cdot \mathrm{d}\vec{\mathbf{S}}$$

where $\mathrm{d}\vec{\mathbf{S}}$ is the unit vector normal to surface S. Recalling that $u_x = u(r)$, we can simplify

$$\dot{M} = \int_0^{2\pi}\int_0^R \rho u(r) r \mathrm{d}r\mathrm{d}\theta \tag{71}$$

$$= \int_0^R 2\pi\rho u(r) r\mathrm{d}r \tag{72}$$

$$= 2\pi\frac{\rho\Delta p}{4\eta L}\int_0^R (R^2 - r^2) r\mathrm{d}r \tag{73}$$

$$= \frac{\pi\rho\Delta p}{2\eta L}\left[R^2\frac{r^2}{2} - \frac{r^4}{4}\right]_0^R \tag{74}$$

$$= \frac{\pi\rho\Delta p}{8\eta L}R^4 \tag{75}$$