Practice session: Astrophysical Hydrodynamics

Kapteyn Learning Community Sirius A
Give motivations and/or derivations for your answers.
Note: Some problems are adapted from past Homeworks or Filippo Fraternali's lecture notes and quiz-type questions.

Part 1: Multiple choice questions

1. Consider a small (non self-gravitating) cold gas cloud $T < 10^4$ K embedded into a vey hot $T > 10^6$ K medium. Which hydrodynamical effect do you expect to dominate the evolution of the cloud?
   (a) Radiative cooling
   (b) Thermal conduction
   (c) Convection
   (d) Hydrostatic equilibrium

2. In a dwarf galaxy, a fraction of interstellar medium is heated at high temperature ($T \approx 10^6$ K) by supenova explosions. Under what condition can we guess that this hot gas can escape the galaxy?
   (a) If it can radiate very efficiently
   (b) If it evolves adiabatically
   (c) If its sound speed is larger than the escape speed from the galaxy
   (d) If its thermal speed is of the order of its sound speed

3. Consider a region of the interstellar medium of a galaxy where star formation is taking place. If we want to describe the gas content using the continuity equation, we need:
   (a) to consider its Lagrangian form
   (b) to add source terms on the right hand side
   (c) to use the equation of state of ideal gases
   (d) to add sink terms on the right hand side

4. The internal energy of a parcel of gas can increase as a consequence of which of the following phenomena? (more than one answer can be correct):
   (a) Advection of the fluid
   (b) Radiative cooling
   (c) Being subject to compression work
   (d) Thermal conduction

5. Consider the accretion disc around a black hole and assume that the black hole fully dominates the gravitational potential. Which of the following statements is correct?
   (a) The disc cannot be in hydrostatic equilibrium
   (b) If the disc is in hydrostatic equilibrium, its thickness is likely constant with radius
   (c) If the disc is in hydrostatic equilibrium, its thickness likely increases with radius
   (d) The disc is self-gravitating

6. As a consequence of the passage of a strong shock wave (shock speed $v_s \approx 1000$ km/s), the interstellar medium of a galaxy is perturbed. Which of the following statements is correct?
   (a) Immediately behind the shock the density grows by orders of magnitude

(b) Immediately behind the shock the temperature grows by orders of magnitude
(c) If the perturbed medium radiates efficiently, the density grows by a factor 4
(d) In adiabatic shocks, the entropy is conserved

7. Consider a shock wave moving at a speed $v_s \approx 500$ km/s through the atomic neutral medium of the Milky Way. Can such a shock ionise the gas?
   (a) Yes, but only if the gas radiates efficiently
   (b) No, but it can produce excitation
   (c) Yes, and that would be called collisionally ionised gas
   (d) No in general, but yes if the gas has a low density

8. At the end of the evolution of a supernova remnant, can some kinetic energy be transferred to the ISM to feed the gas turbulence?
   (a) Yes, nearly all the initial energy of 10^{51} erg, as a consequence of momentum conservation
   (b) Yes, but only about half of the initial energy due to energy equipartition
   (c) Yes, but only a small percentage of the initial energy, due to radiative losses
   (d) No, all the initial energy is converted into heat

9. According to the Sedov solution of a blast wave, the shock propagates through the medium in which of the following ways?
   (a) It accelerates pushed by the high pressure in the internal bubble
   (b) It decelerates as it perturbs more and more initially unperturbed gas
   (c) It expands at nearly constant speed, which is about 3/4 of the initial shock speed
   (d) The regimes of acceleration or deceleration depend on the cooling time in the medium right behind the shock wave

10. Consider a supernova remnant produced by a standard supernova explosion (normal energy input), but taking place in an underdense region of the ISM. Which of the following statements is correct? (More than one answer can be correct)
   (a) The cooling time of the perturbed medium is shorter and the Sedov phase lasts longer
   (b) The cooling time of the perturbed medium is larger and the Sedov phase last longer
   (c) The radius grows in time during the radiative phase with the same dependence (slope) it would have in the normal ISM
   (d) The supernova remnant radiates away all its initial energy

11. Consider a cold cloud in the interstellar medium nearly at constant temperature and having a mass two orders of magnitude smaller than the Jeans mass, calculated given its temperature and density. What can we conclude about its dynamical state?
   (a) The cloud is gravitationally unstable and it should collapse in a free-fall time
   (b) The cloud is gravitationally unstable, but it can be stabilised by turbulence or magnetic fields
   (c) The cloud is not gravitationally unstable, but it can become convectively unstable
   (d) The cloud must be pressure confined

12. Consider a portion of warm, $T \approx 10^4$ K, ISM in the path of a hot, $T \approx 10^6$ K, wind produced by massive stars in its vicinity. The wind is moving at fast speeds: hundreds of km/s. What kind of instability will likely develop at the surface of the warm gas?
   (a) Rayleigh-Taylor instability
   (b) Kelvin-Helmholtz instability

(c) Thermal instability
(d) Difficult to say without calculating the Jeans mass of the hot gas

13. The region of space around galaxies is permeated by rarefied gas at a few million degrees, but some observations have also shown the presence of clouds of relatively cold $T \approx 10^4$ K gas. These clouds have masses much smaller than the Jeans mass at their temperature and density. Their lifetime is estimated to be quite long, at least a few hundred Myr. What are the effects that can help them survive? (More than one answer can be correct)
    (a) Self-gravity
    (b) Radiative cooling
    (c) Pressure confinement from the external medium
    (d) Convection

14. Which of the following effects can occur as a consequence of thermal instability?
    (a) A portion of gas in the hot halo of a galaxy initially in hydrostatic equilibrium can cool radiatively and fall towards the centre of the potential well
    (b) Gas at intermediate (unstable) temperatures can be found in interface regions, between hot and cold gas
    (c) The interior of a star can become unstable and release large amount of radiation
    (d) The various phases of the ISM can reach pressure equilibrium

15. The presence of viscosity in a fluid introduces important differences with respect to an ideal (inviscid) fluid. Which of the following statements are correct? (More than one answer can be correct)
    (a) In the presence of viscosity some kinetic energy is turned into internal energy
    (b) If shear viscosity is present, Kelvin-Helmholtz instability will develop more rapidly
    (c) Bulk viscosity is only important in incompressible fluids
    (d) The Euler equation is no longer valid and we need to add some viscous terms

16. Which of these statements about the Kolmogorov theory of turbulence and its applications is correct? (More than one answer can be correct)
    (a) Turbulent entities, eddies, are fully developed at all possible scales down to the dissipation scale
    (b) Energy must be continuously injected in the small scales and transferred to the largest ones
    (c) If the medium is compressible, the energy dissipation will be always more efficient than Kolmogorov predicts
    (d) In the largest eddies, viscosity must be important

17. Consider a molecular cloud with the presence of an internal magnetic field. What effect can this magnetic field have on the dynamics of the cloud?
    (a) It can enhance the radiative cooling
    (b) It can counteract gravity through magnetic pressure
    (c) It can facilitate the collapse through emission of Alfvén waves
    (d) A magnetic field cannot be important in a molecular clouds because there are no free electrons

Part 2: Problems

1. **What is a fluid anyway?**
Hydrodynamics deals with the physics of fluids, but what is a fluid? And can we always use these concepts?
   (a) Describe what a fluid is in the context of Astrophysical Hydrodynamics.
   (b) Discuss and compare the relevant length scales.
   (c) Explain why the fluid approximation is good for the solar interior, but will not be a good description for the solar wind. Consider that the mean free path inside the Sun is of the order of 10^{-4} cm and can be greater than 1 AU for the solar wind.

2. **Lagrangian and Eulerian view on life**
What is the Eulerian description of fluid dynamics, and what is the Lagrangian description. Describe the physical meaning and difference between Eulerian derivative of a thermodynamic quantity Q,
$$\frac{\partial Q}{\partial t} \tag{2}$$
and the Lagrangian derivative
$$\frac{DQ}{Dt} \tag{3}$$
and state the relationship between these two derivatives for a fluid with flow field $\vec{v}$.

3. **The symbols... what do they mean?**
Consider the following equation of hydrodynamics:
$$\rho T \frac{D\mathcal{S}}{Dt} = -\boldsymbol{\nabla} \cdot \mathbf{q} - \rho \mathcal{L} \tag{5}$$
   (a) Describe what it represents and what is the quantity $\mathcal{S}$.
   (b) What do the two terms on the right hand side represent?
   (c) How can the $\mathcal{L}$ term be further developed?

4. **Star goes boom**
When a massive star reaches the end of its life, it will violently blast a large part of itself into outer space. Indeed, these are one the most violent events in the universe. The evolution of the supernova remnant, as it propagates through the ISM, can be described in terms of three phases.
   (a) Which are these three phases, and what is the time evolution of the supernova remnant shell radius r for the first two phases?
   (b) Discuss the physics of the different stages, and the transitions between them.

5. **More Supernova explosions!**
In this exercise we are going to have a deeper look at the physics of supernova remnant Shock Waves and what effect it has on the surrounding medium.
   (a) Write down the Rankine-Hugoniot jump equations
   (b) Explain the origin of each of the equations
   (c) The location of the supernova implies that it happens in an isothermal environment. Derive the compression factor between the initial and final medium and find the solution for the velocities. (Tip: use that the pressure becomes $P(\rho) = \rho c_s^2$, with c_s the speed of sound)

6. **Space tornadoes and Kelvin's circulation theorem**
In the midst of chaos you find yourself pondering about a potential danger. No one in your spaceship remotely has the intellectual capacity to even fathom such a crisis. As the ship drifts through turbulent and dense ISM, and your colleagues fall asleep, you find yourself determined to recover the tool that may give insight to possible impending doom. *Space tornadoes.*

(a) For a fluid with flow velocity field $\vec{\mathbf{u}}$, write the Eulerian description of the Euler equation, taking into account the influence of the gravity field $\vec{\mathbf{g}}$ and the gas pressure P. Ignore viscous effects.

(b) Show that for a *barotropic* flow in a conservative gravitational field (i.e. $\vec{g} = \vec{\nabla}\phi$) we have:

$$\frac{\partial \vec{\omega}}{\partial t} + \boldsymbol{\nabla} \times (\vec{\omega} \times \vec{\mathbf{u}}) = 0 \tag{11}$$

where $\vec{\omega}$ is the *vorticity*.
Hint: consider the identities

$$\boldsymbol{\nabla}\frac{1}{2}a^2 = (\vec{\mathbf{a}} \cdot \boldsymbol{\nabla})\,\vec{\mathbf{a}} + \vec{\mathbf{a}} \times (\boldsymbol{\nabla} \times \vec{\mathbf{a}}) \tag{12}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \vec{\mathbf{a}}) = \boldsymbol{\nabla}\,(\boldsymbol{\nabla} \cdot \vec{\mathbf{a}}) - \nabla^2\vec{\mathbf{a}} \tag{13}$$

(c) You define the flow's *circulation* Γ around a circuit C (with surface area A) by

$$\Gamma := \oint_C \vec{\mathbf{u}} \cdot d\vec{\mathbf{l}} = \int_A \vec{\omega} \cdot d\vec{\mathbf{A}}, \tag{14}$$

how does Γ change with time?
Use that:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \int_A \frac{\partial \vec{\omega}}{\partial t} \cdot d\vec{\mathbf{A}} + \oint (\vec{\omega} \times \vec{\mathbf{u}}) \cdot d\vec{\mathbf{l}} \tag{15}$$

The resulting relation is *Kelvin's circulation theorem*. What does it imply for inviscid barotropic space tornadoes and should you and your crew be scared?

7. **Atmospheres of galaxy clusters**

Galaxy clusters are the largest and most massive structures in the Universe. Their main mass component is dark matter but they also have hot gas, while the stellar component in galaxies can be considered negligible. Here, we assume that the gravitational potential is fully dominated by the dark matter and can be written as

$$\Phi(r) = v_c^2 \ln\left(\frac{r}{r_0}\right) + \Phi_0, \tag{28}$$

where v_c is a parameter (the circular speed), r_0 is a reference radius and Φ_0 is the value of the potential at that radius.

(a) Assume that the hot gas has an isothermal equation of state ($P = c_s^2\rho$, with c_s constant) and find its radial density profile. How does this profile change if we change the gas temperature?

(b) Assuming that the gas temperature is $T = 10^8$ K, derive the cooling time as a function of r. Estimate (approximately) the value of the *cooling radius* (distance from the centre of the cluster at which the cooling time is equal to the Hubble time). For this calculation you can use $r_0 = 100$ kpc, $n_{\mathrm{e},0} = 10^{-3}\,\mathrm{cm}^{-3}$ (electron density at r_0) and $v_s = \sqrt{2}c_s$. The cooling function of collisional ionisation equilibrium at $T = 10^8$ K is $\Lambda(10^8\,\mathrm{K}) = 2.5 \times 10^{-23}\,\mathrm{erg\ cm^3\ s^{-1}}$.

(c) Assume then that the gas has, instead, a polytropic equation of state: $P = A\rho^\beta$ (with A and β constant parameters, $\beta > 1$), and determine the density profile in this case. Is the temperature constant or a function of r in this case?

8. **Navier and Stokes in a secret tunnel**

We are going to investigate the Navier-Stokes equation, and its application towards the flow through a cylindrical pipe. The Navier-Stokes equation is an extension, and a more complete, version of the Euler equation now also including an internal friction term. the Navier-Stokes equation is given by:

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \eta\nabla^2\mathbf{u}. \tag{49}$$

One situation in which we can derive an analytical expression for a viscous flow is that of a steady, laminar flow through a pipe (or secret tunnel, through the mountain) of radius R. Steady means that the time derivative of the relevant physical quantities is zero $\partial/\partial t = 0$. For the pipe we use cylindrical coordinates, x along the length of the pipe, radius r and sectional angle θ.

The flow is only along the length of the pipe, $u_x = u$. The radial and angular components of the fluid velocity are zero: $u_r = u_\theta = 0$. The flow is axisymmetric, and can only vary in the radial direction or, possibly, along the x-direction, that is $\partial/\partial\theta = 0$. The flow is fully developed and will not vary along the x-direction as the diameter of the pipe is constant along its length: $\partial u/\partial x = 0$

(a) Show that the Navier-Stokes equation for this situation can be written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\eta}\frac{\partial p}{\partial x}, \tag{50}$$

$$0 = \frac{1}{\eta}\frac{\partial p}{\partial r}. \tag{51}$$

(b) Argue that the pressure drop along the x-direction in the pipe is linear, i.e. that $(\partial p/\partial x)$ = constant, and thus,

$$\frac{\partial p}{\partial x} = -\frac{\Delta p}{L}. \tag{52}$$

Where Δp is the pressure drop along a length L of the pipe.

(c) Subsequently, show that the general solution for this equation can be written as,

$$u = -\frac{1}{4\eta}\frac{\Delta p}{L}r^2 + c_1 \ln r + c_2. \tag{53}$$

Where c_1 and c_2 are integration constants.

(d) From the boundary condition that the velocity v is finite at $r = 0$, it follows that $c_1 = 0$. Show that from the no slip boundary condition at the wall of the pipe, i.e. $u_x = u = 0$ at $r = R$, that,

$$c_2 = \frac{1}{4\eta}\frac{\Delta p}{L}R^2. \tag{54}$$

(e) And that therefore the generic solution for the flow field is given by,

$$u(r) = \frac{1}{4\eta}\frac{\Delta p}{L}(R^2 - r^2). \tag{55}$$

This is called the Hagen-Poiseuille equation. Describe and explain how the flow field $u(r)$ in the pipe behaves as function of radius r.

(f) Show that for a fluid with a density ρ the amount of mass transported through a surface of constant x per unit time is given by

$$\dot{M} = \int_0^R 2\pi\rho u r \mathrm{d}r = \frac{\pi\rho\Delta p}{8\eta L}R^4. \tag{56}$$