Practice session: Planetary Systems

Kapteyn Learning Community Sirius A
Give motivations and/or derivations for your answers.

Calculators are allowed.
This practice session is intentionally slightly longer than an average exam.

1. **Dynamics**

   (a) Explain briefly the concepts of Lagrangian points, Hill sphere and Roche limit.

   (b) Explain the Yarkovsky Effect. Say it is working on a single asteroid in the Main Asteroid Belt. How will its orbit be affected by it, if it is orbiting *and* spinning counterclockwise?

   (c) The semi-major axis of Jupiter is 5.205 AU. What are the semi-major axes of the 3:2, 4:3, and 3:1 mean motion resonances?

   (d) How do these mean motion resonances relate to the structure of the Main Asteroid Belt?

Solution:

(a) The **Lagrangian Points** are the points where, in a restricted three-body problem with one large body and a smaller – roughly circularly – orbiting body, test particles in a rotating would feel no net force.

The **Hill sphere** is the approximate limit to a body's gravitational sphere of influence.

The **Roche limit** is the boundary within which a moon's self-gravity is smaller that the tidal forces exerted by the body it orbits, causing the moon to be ripped apart eventually. Inside Roche limits we find planetary rings.

(b) The Yarkovsky effect is caused by a difference in surface temperature of the morning and afternoon side of a sufficiently large body, causing the hotter afternoon area to exert a slightly larger radiation pressure on the body on their side. For a body with a prograde rotation (as described in the problem) the orbit is expanded slightly over a long time, as radiation pressure increases the object's orbital velocity slightly, vise versa with a retrograde rotation.

(c) As:

$$P_{\mathrm{yr}}^2 = a_{\mathrm{AU}}^3$$

Then:

$$\left(\frac{P_{\text{resonance}}}{P_{\text{Jupiter}}}\right)^2 = \left(\frac{a_{\text{AU, resonance}}}{a_{\text{AU, Jupiter}}}\right)^3$$

$$a_{\text{AU, resonance}} = \sqrt[3]{\left(\frac{P_{\text{resonance}}}{P_{\text{Jupiter}}}\right)^2} \cdot 5.205\,\text{AU}$$

$$a_{3:2} = \left(\frac{2}{3}\right)^{\frac{2}{3}} \cdot 5.205\,\text{AU} \approx 3.972\,\text{AU}$$

$$a_{4:3} = \left(\frac{3}{4}\right)^{\frac{2}{3}} \cdot 5.205\,\text{AU} \approx 4.297\,\text{AU}$$

$$a_{3:1} = \left(\frac{1}{3}\right)^{\frac{2}{3}} \cdot 5.205\,\text{AU} \approx 2.502\,\text{AU}$$

(d) The mean motion resonances with Jupiter (and other major planets) help to explain the existence of *Kirkwood gaps* in the orbital distribution of asteroids in the Main Asteroid Belt, where asteroids were cleared from the Belt. Yet the resonances can also stabilize certain groups of asteroids' orbits leading to *groups* like the Hildas and Thules.

2. Meteorites

(a) Explain the difference between differentiated and undifferentiated bodies.

(b) Why does this matter for the classification of meteorites?

(c) See figures 1 and 2 below, what kind of meteorites are displayed? Motivate your answer.



Figure 1: A nice rock



Figure 2: Yet another nice piece of rock

(d) Describe methods about how we can determine the size, shape and albedo of asteroids.

Solution:

(a) Differentiated bodies have undergone a period of time where they were mostly molten and thus the iron (and other heavy elements) sank to the center. Undifferentiated bodies have not undergone melting, but might have still undergone other processes that changed them.

(b) Differentiation is a density-based phase seperation. This means that meteorites originating from a differentiated body have very different compositions than meteorites originating from undifferentiated bodies.

(c) **Figure 1**: This is a chondrite which can be seen from the dark fine-grained matrix covering most of the rock.
Figure 2: This is an iron meteorite. This can be seen by the Windmanstätten pattern.

(d) Size and albedo can be determined using radiometry which means combining measurements at visible and infrared wavelengths. The shape can be determined over a longer period of time by measuring lightcurves of the asteroid from many angles. The shape can also be determined with radar observations.

3. Comets

(a) Describe the compositional and orbital differences between asteroids and comets.

(b) In figure 3 you can see observable properties of three components, of which one are silicates, of a comet. What are the origins of the components 1 and 2?

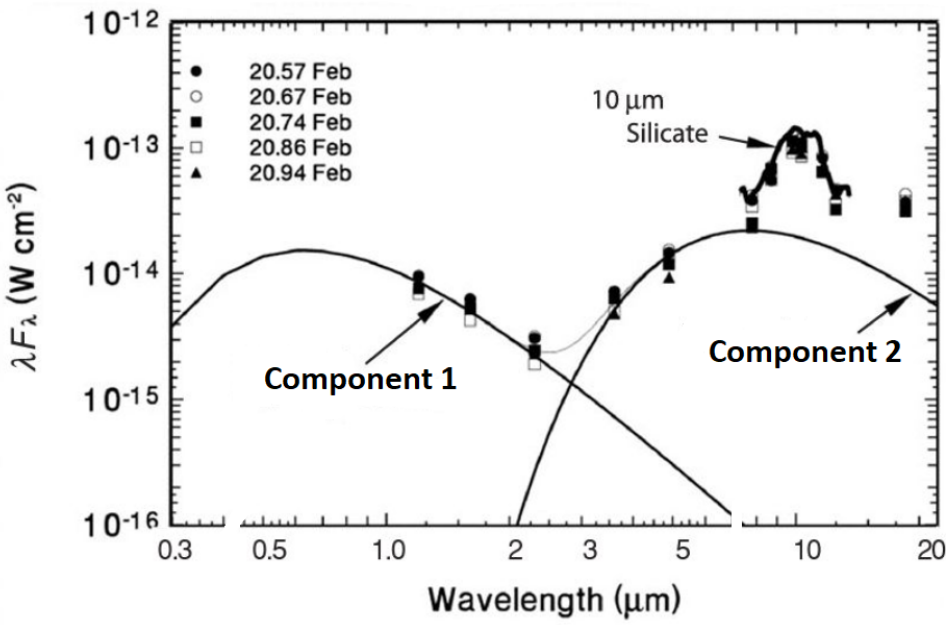


Figure 3: Power per area vs. wavelength of three components of a comet

(c) Describe two methods that are used to find exocomets.

Solution:

(a) Asteroids are rocky planetary bodies that orbit the sun with low eccentricity orbits with a radius smaller than or equal to Jupiter. Comets contain a solid core and a large gas coma with an even larger hydrogen cloud and when getting relatively close to the sun it has a dust and ion tail trailing it. Comets usually travel on extremely eccentric orbits around the sun.

(b) Component 1 is from the dust tail trailing the comet, which reflects the light of the Sun and therefore mimics the black-body emission of the Sun.
Component 2 is from the thermal emission of the coma.

(c) **Spectroscopy**: A time series spectroscopy can be used to find exocomets. Variations in the absorption lines of a star can be seen when an exocomet passes its star due tue the dust tail and gas cloud surrounding the comet.
Transit photometry: As a comet passes close to its star, it blocks a portion of the light. Not only the nucleus but also the dust trail can block some light. In a photometry spectrum this can be seen as a dip in light.
Sub-mm emission from volatiles: As the comet passes close to the sun it gets heated up which means the volatile elements of the comet become active and radiate sub-mm radiation.

4. **Atmospheres**

(a) What is the difference between eddy diffusion and molecular diffusion in planetary atmospheres?

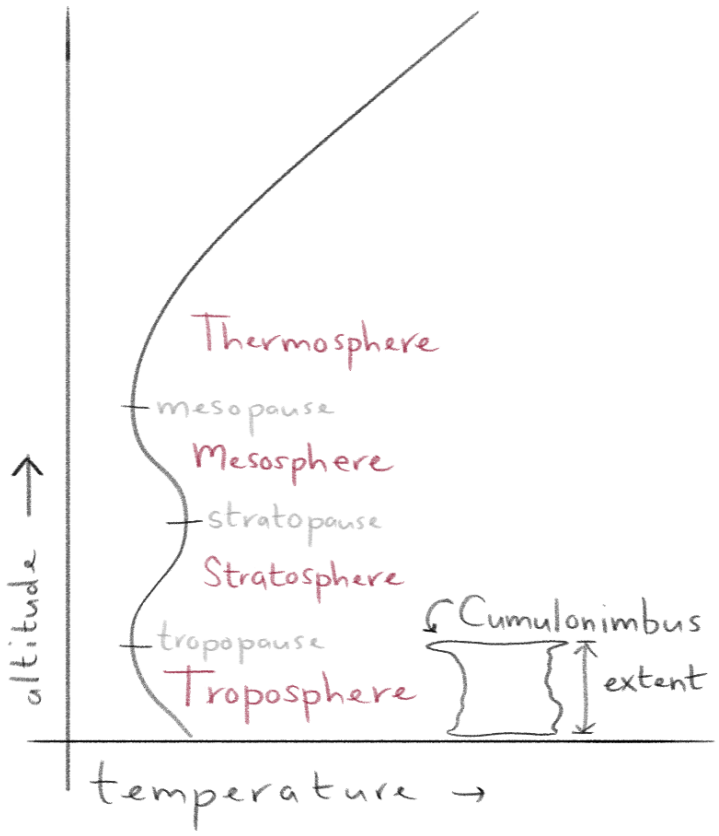
(b) Sketch the thermal profile of Earth and indicate the relevant atmospheric layers. What causes the upper layer to increasingly get hotter with altitude to temperatures at or above 800 K?

(c) In your sketch, indicate roughly the maximum height extent of a cumulonimbus cloud formation (if you want you can also sketch its shape). How does this relate to the thermal profile?

Solution:

(a) **Molecular diffusion** is the process by which the individual thermal motion of molecules resulting in a gradual mixing of different atmospheric constituents.

Eddy diffusion is the result of macroscopic turbulent motion in the atmosphere causing constituents to mix — it dominates in the higher density part of the atmosphere below the homopause (at ~ 100 km).



(b) In the *thermosphere* molecules are heated radiatively by absorbing solar (UV) radiation. The increasingly lower densities make it harder for molecules to collide and loose this thermal energy through infrared radiative cooling, causing the temperature to rise with altitude to the high temperatures at or above 800 K.

(c) (See the figure above)

The inversion of the temperature gradient at the tropopause causes the upward convection of air to halt, putting a limit on the maximum vertical extent of the cloud-layer on Earth (This explains the anvil shape of big cumulonimbus clouds as the air starts spreading horizontally outward from the central convective column)

5. **Planetary Formation**

(a) How do µm to ∼ cm sized particles grow in a protoplanetary disk? Explain how the so-called centimeter barrier then arises.

(b) Skipping the centimeter barrier, after planetesimals reach ∼ km size, how do they continue

to grow? And when do they stop growing?

Solution:

(a) Particles smaller than $\sim$ cm-size grow in size by cohesion, colliding and then sticking together. When particles approach $\sim$ cm in size however, they tend to break apart as much as during collisions as they cohesively stick together. This causes the process of growth beyond this size to be poorly understood, leading to the *centimeter barrier*.

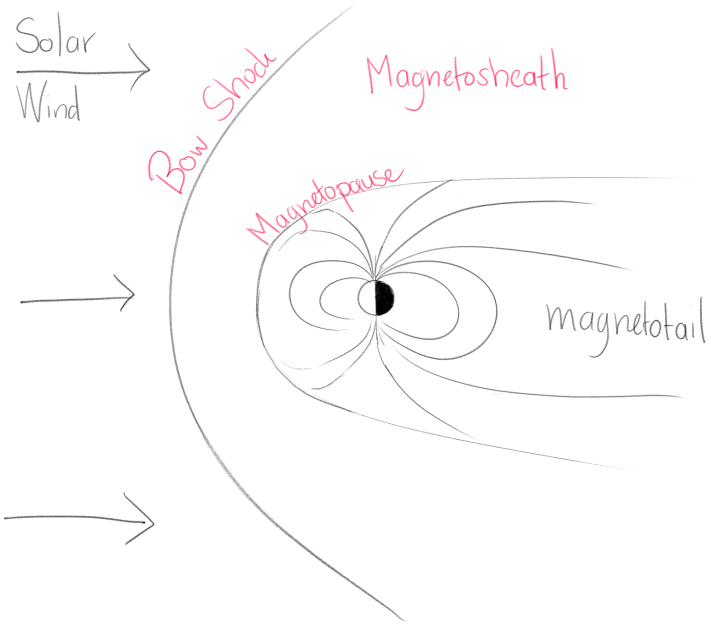
(b) Sufficiently sized planetesimals start to accrete smaller planetesimals through gravitational interactions, favouring the largest planetesimals, until eventually a few larger bodies remain. When the largest initial bodies have reached their *isolation mass* — having accreted all available smaller bodies — they stop growing

6. **Magnetospheres**

(a) Make a sketch of a typical planetary magnetosphere with the incoming solar wind. In particular, indicate the bow shock, the magnetopause, and the magnetosheath. Explain in one sentence what these three terms are.

(b) How is the bouncing motion of charged particles in the magnetosphere caused? What effect can this have on a planetary atmosphere?

Solution:



(a)

The **Bow Shock** is boundary at which a planetary charged particle population's interaction with Solar wind causes it to start moving subsonically instead of supersonically,

causing an increase in plasma density. The region downstream of the Bow Shock is the **Magnetosheath**. The **Magnetopause** is the boundary of the planetary magnetosphere and the place where the solar wind ram pressure is equal to the magnetospheric pressure ($P_{\mathrm{SW}} = P_{\mathrm{m}}$).

(b) The bouncing motion along the field lines is caused by the charged particle moving along the field line (with kinetic energy conserved) towards the planetary surface where field strengths start increasing, until the field-aligned Lorentz force forces the particle's velocity along the field line to become zero and flip direction. The bouncing motion along the field lines causes the aurora in the planetary atmosphere.

7. Planetary surfaces

(a) Secondary craters are formed by the ejected material of an impact event. It has been found that these secondary craters are further away from the initial impact crater on the Moon than Mercury. Why is this the case?

(b) What are two differences between a simple crater and a complex crater?

Solution:

(a) Mercury has a greater surface gravity than the moon. This means that rocks and debris excavated from the primary craters fall down faster again and thus land closer to the primary crater.

(b) Simple craters are often small (up to a few km in diameter) and have a simple bowl-like shape. Complex craters are usually larger (up to few hundred km) and have a flat floor with a central peak. Complex craters also have collapsed rims which result in terraces. So the two differences are in size and shape.

8. Planetary interiors

(a) Use the equation of hydrostatic equilibrium to find the central pressure of a planet with constant density. Also find the expression for the pressure as a function of distance from the center of the planet.

(b) In reality, the density is not a constant. How does the density really evolve and what effect will this have on the central pressure you found above (higher/lower), explain.

Solution:

(a) The equation for hydrostatic equilibrium is given as follows:

$$\frac{dP}{dr} = -g_p(r)\rho(r) \tag{1}$$

From this we can find an expression for pressure as function of distance r from the center of the planet. We do this as follows:

$$dP = -g_p(r)\rho(r)dr$$
$$P(r) = -\int_R^r g_p(r)\rho(r)dr$$

This is the general expression for the pressure P as function of distance r from the center of the planet. For a planet with constant density, $\rho(r) = \rho$:

$$P(r) = -\int_R^r g_p(r)\rho dr$$
$$g_p(r) = \frac{GM(r)}{r^2}$$
$$M(r) = V(r)\cdot\rho = \frac{4\pi r^3\rho}{3}$$
$$\begin{aligned} P(r) &= -\int_R^r G\frac{4\pi r^3}{3}\frac{1}{r^2}\rho^2 dr \\ &= -\frac{4\pi G\rho^2}{3}\int_R^r r dr \\ &= -\frac{4\pi G\rho^2}{3}(\frac{1}{2}r^2 - \frac{1}{2}R^2) \\ &= -\frac{2\pi G\rho^2}{3}(r^2 - R^2) \end{aligned}$$

So the general expression for the pressure at radius r of a planet with constant density is $P(r) = \frac{2\pi G\rho^2}{3}(R^2 - r^2)$.
For the central pressure (r = 0) this then becomes:

$$P(0) = -\frac{2\pi G\rho^2}{3}(0 - R^2) = \frac{2\pi G\rho^2}{3}R^2$$
$$\rho = \frac{M}{V} = \frac{3M}{4\pi R^3}$$
$$\rho^2 = \frac{9M^2}{16\pi^2 R^6}$$
$$P(0) = \frac{2\pi G}{3}\frac{9M^2}{16\pi^2 R^6}R^2$$
$$P_c = \frac{3GM^2}{8\pi R^4}$$

(b) Generally the density increases towards the center of the planet. This means that the actual central pressure will often be larger than the previously obtained estimate. As more mass is closer to the center, more gravitational force is exerted since gravity is inversely proportional to the distance squared.

9. **General Questions**

Answer the following questions in a few sentences.

(a) What is the IAU definition of a planet?

(b) What is the strongest evidence for planetary migration? Explain.

(c) Describe The Nice Model and The Grand Tack Hypothesis.

(d) What is the Rayleigh number and what does it depend on?

(e) What is the oxygen fugacity (f_{O_2})? What do IW-2 and IW+2 represent, and what is their effect on a secondary atmosphere?

(f) What is the adiabatic lapse rate and what is the difference between dry and moist?

(g) What is a Hadley cell and how does it form?

(h) Describe the two following atmospheric escape processes: Jeans (include the definition of the exobase and the Jeans escape parameter) and Hydrodynamic (mention and explain the parameters that affect the mass-loss rate).

Solution:

(a) 1. It is in orbit around the Sun.
2. It has sufficient mass for its self-gravity to overcome rigid body forces, so that it assumes a hydrostatic equilibrium (nearly round) shape.
3. It has cleared the neighbourhood around its orbit.

(b) Hot Jupiters, which are gas giant exoplanets orbiting their host stars with periods of just a few days. They cannot have formed in their location since there is not enough material, and temperatures are too high for solids, so they must have formed further out and moved inward, either by disk migration or by later high-eccentricity migration followed by tidal circularisation.

(c) The Nice Model: giant planets formed in a more compact configuration than today and underwent a dynamical instability. The Jupiter-Saturn period ratio changed discontinuously from 1.6 to 2.5, which suggests the existence of a fifth giant planet that was scattered off Jupiter and ejected during the instability (jumping-Jupiter scenario). Uranus and Neptune scattered outwards to their current orbits, the primordial Kuiper Belt was disrupted, and delivered impactors throughout the inner solar system.

The Grand Tack: Jupiter first migrated inward to AU via Type II migration, then reversed course when Saturn caught up and became trapped in the 3:2 mean-motion resonance with Jupiter (combined torques from their mutual interaction with the disk

drove both planets outward). This resulted in scattered and depleted material in the inner solar system, potentially explaining the small mass of Mars and the compositional structure of the asteroid belt.

(d) The Rayleigh number quantifies the balance between convection and viscous resistance of a fluid layer. Buoyancy arises when there is a density difference between the cooler upper and hotter lower parts of the layer, but viscosity and thermal diffusion (how fast heat can transport through the fluid) resist convection. Convection dominates if the Rayleigh number exceeds a critical value. It is proportional to the thermal expansion coefficient (how much the fluid expands when heated), the density of the fluid, the gravitational acceleration (gravity is always required for convection), the temperature difference across the layer, and the layer depth. It is inversely proportional to the thermal diffusivity and the dynamic viscosity.

(e) The oxygen fugacity (f_{O_2}) measures the chemical availability of oxygen in a magma ocean, represented as the effective partial pressure of O_2 in equilibrium with the melt. Values are quoted relative to the iron-wüstite (IW) buffer, $2Fe + O_2 \rightleftharpoons 2FeO$. IW-2 indicates 2 orders of magnitude below reference corresponding to a reducing environment for which the iron remains metallic (outgassed secondary atmosphere dominated by $H_2, CO.N_2$). IW+2 indicates 2 orders of magnitude above reference (oxidising environment) for which the iron oxidises to FeO and enters the silicate (outgassed secondary atmosphere dominated by $CO_2, H_2O.N_2$).

(f) The adiabatic lapse rate is the rate of temperature decrease as a parcel of air rises adiabatically (without any heat exchange with its surroundings) through the atmosphere due to convection (units of Kelvin over length). A dry adiabatic lapse rate represents a parcel of air without water which contrasts a moist adiabatic lapse rate. For the latter, condensing water vapour adds more heat to the parcel (latent heat), reducing the cooling rate, and resulting in a shallower profile compared to the dry.

(g) A Hadley cell is the wind loop formed in Earth's atmosphere between the equator and the subtropics. Intense solar heating at the equator warms the surface and the air above it, resulting in warm, buoyant air rising at the equator, creating a low-pressure zone. At altitude, the air flows toward higher latitudes (poles). As it moves to the poles, it cools radiatively, becomes denser, and sinks at 30° latitude (the subtropics), creating high-pressure zones (where Earth's great deserts are located). At the surface, air flows back toward the equator to replace the rising air.

(h) Jeans escape: gas molecules in the atmosphere have a thermal velocity described by the Maxwell-Boltzmann distribution (most probable $v_{\text{th}} = \sqrt{2k_B T/m}$). While the most probable speed is too low, some molecules at the tail-end of the distribution have a speed greater than the gravitational escape speed beyond the exobase, $v_{\text{esc}} = \sqrt{2GM_P/r_{\text{exo}}}$. The exobase is defined as the point in the atmosphere where the mean free path is equal to the pressure scale height. Beyond that, molecule motions become ballistic with no collisions, allowing escape to happen. The Jeans escape parameter is the ratio of gravitational binding energy to thermal energy for a molecule at the exobase. When $\lambda_J \gg 1$, very few molecules have enough energy to escape and escape is slow. When $\lambda_J < 2$–3, a

substantial fraction of molecules can escape, and the atmosphere erodes rapidly.

Hydrodynamic escape: bulk hydrodynamic outflow escape instead of particle-to-particle due to exposure to intense extreme ultraviolet (EUV) radiation from a young, active star. The mass-loss rate is derived by equating the incident stellar radiation to the gravitational binding energy of the leaving gas. It is proportional to the EUV flux and the radius of the planet cubed (extended, low-density envelope more prone to escape). It is inversely proportional to the planet's mass since higher mass means higher surface gravity and a tighter grip around the atmosphere.