Practice Session: Formation and Evolution of Galaxies

Kapteyn Learning Community Sirius A

The following problems are taken and adapted from Filippo Fraternali's course material (with his permission). The session is intentionally longer than the exam.

1. **Multiple-choice Questions**
   Chose the correct answer from the following questions. Multiple answers can be correct.

   1. The primordial power spectrum of fluctuations is modified by the transfer function, $T(k)$. Why is this needed?
      (a) Because the Jeans mass of the dark matter component is too low before recombination.
      (b) Because perturbations that enter the horizon before matter–radiation equality stop growing.
      (c) Because perturbations that enter the horizon between equality and recombination stop growing.
      (d) Because superhorizon perturbations cannot grow.

   2. Which of these statements regarding gas accretion onto dark matter halos are corect?
      (a) A gas component inside a dark matter halo will always be at the virial temperature
      (b) The gas accreting onto halos with masses above the critical mass is shocked to the virial temperature
      (c) Gas accretion onto halos with masses below the critical mass does not shock heat
      (d) Gas accretion at a temperature below the virial temperature is virtually impossible

   3. Under what conditions does the instantaneous recycling approximation work well in chemical evolution models?
      (a) When we are dealing with α-elements
      (b) When we are dealing with elements like Fe and others produced by supenova Ia
      (c) When the retun fraction is below 0.4
      (d) When the time that a star spends in red giant, horizontal and asymptotic giant branches can be considered short

   4. Which of these statements on supernova feedback are correct?
      (a) Supernovae that take place inside an O/B association end up being less efficient in feeding the ISM with kinetic energy
      (b) A super bubble can eject gas outside the discs of galaxies
      (c) If the outflow speed exceed the escape speed, the gas has the potential to leave the dark matter halo hosting the galaxy
      (d) Supernova feedback is critical to explain the shape of the galaxy stellar mass function at the high-mass end

Solution:

1. (b): before matter-radiation equality, the Universe is radiation-dominated. Dark matter perturbations entering the horizon during this era grow very slowly, so their growth is suppressed relative to perturbations entering later. The transfer function $T(k)$ describes this scale-dependent suppression of the primordial power spectrum. (a) is incorrect because cold dark matter has negligible pressure, so its Jeans mass is not the main cause. (c) is incorrect because after equality, matter perturbations can grow during the matter-dominated era. (d) is incorrect because superhorizon perturbations can evolve; they are not simply unable to grow.

2. (b) and (c): above the critical halo mass, accreting gas typically undergoes a stable virial shock and is heated to approximately the virial temperature. Below the critical mass, gas can accrete through cold flows without being shock-heated to the virial temperature. (a) is incorrect because its temperature depends on its accretion history, cooling, and other physical processes. (d) is incorrect because accretion below the virial temperature is still possible; cold-mode accretion is a common example. The critical mass separates regimes where stable virial shocks can be maintained from regimes where gas can flow in relatively cold.

3. (a): alpha elements (e.g., O, Mg, Si) are mainly produced by massive stars, which have short lifetimes. Their enrichment can therefore be approximated as instantaneous compared with galactic evolution timescales. (b) is incorrect because Type Ia supernovae produce much of the iron after significant time delays, (c) is incorrect because the return fraction does not determine whether the approximation is valid, and (d) is incorrect because the approximation depends on whether stellar lifetimes are short compared with the timescale of galactic evolution, not specifically on the time spent in the red giant, horizontal, and asymptotic giant branches.

4. (b) and (c): (a) is incorrect because supernovae within an O/B stellar association can combine their energy to create a superbubble, making feedback more effective at driving large-scale gas motions rather than less efficient. (d) is incorrect because supernova feedback is especially important for regulating star formation in low- and intermediate-mass galaxies. The high-mass end of the stellar mass function is primarily regulated by processes such as active galactic nucleus (AGN) feedback.

2. Surface Brightness of a Nearby Galaxy

The surface brightness (I) of an extended astronomical source can be defined as the source's luminosity (L) per unit area (S):

$$I = \frac{dL}{dS}. \tag{1}$$

(a) By using the relation between luminosity and flux, show that the surface brightness is independent of distance for nearby galaxies (not at cosmological distances).

(b) Consider the observation of a nearby galaxy in the B-band that gives us its surface brightness at the effective radius $I_{e,B} = 22$ mag arcsec^{-2}. Calculate the corresponding surface brightness in units of $L_{\odot,B}$ pc^{-2}. For this calculation, you can use a Solar absolute magnitude in the B-band of $M_{\odot,B} = 5.31$.

(c) Assume that the surface brightness profile of the same galaxy can be described by an exponential function of the form

$$I(R) = I_0 e^{-R/R_d}, \tag{2}$$

with I_0 the central surface brightness and R_d the scale length. If the measured effective radius and distance to the galaxy are, respectively, $R_e = 10''$ and $d = 5$ Mpc, calculate the absolute magnitude of the galaxy in the B-band, M_B.

Tip 1: To calculate the total luminosity, you can simply integrate the surface brightness out to $R = \infty$.

Tip 2: Find the relation between R_e and R_d.

(d) In the direction of this galaxy, the average column density of hydrogen in the Milky Way is $N_H = 4 \times 10^{20}$ cm^{-2}. Use it to estimate the extinction in the B-band and update the value of M_B. Given the information you have obtained about this galaxy, how does it compare with the Milky Way?

Tip: To estimate the extinction you can use the definition of the slope of the extinction curve:

$$R_V \equiv \frac{A_V}{A_B - A_V}, \tag{3}$$

with $R_V = 3.1$, an average value in the Milky Way.

Solution:

(a) The relation between luminosity and flux is

$$F = \frac{L}{4\pi d^2}. \tag{4}$$

For a small area element, we have

$$dF = \frac{dL}{4\pi d^2} \Rightarrow dL = dF\, 4\pi d^2 \tag{5}$$

The solid angle subtended by this area element is

$$d\Omega = \frac{dS}{d^2} \Rightarrow dS = d\Omega\, d^2 \tag{6}$$

Replacing the above:

$$I = \frac{dL}{dS} = \frac{dF\,4\pi d^2}{d\Omega\,d^2} = \frac{dF}{d\Omega}\frac{1}{4\pi} \tag{7}$$

The distance d cancels out, showing that the observed surface brightness is independent of distance for nearby galaxies (depends on the solid angle).

(b) The surface brightness in magnitudes per square arcsecond is related to the luminosity surface density by

$$\mu_B = M_{\odot,B} + 21.572 - 2.5\log_{10}\left(\frac{I_B}{L_{\odot,B}\,\mathrm{pc}^{-2}}\right). \tag{8}$$

Rearranging gives

$$\frac{I_B}{L_{\odot,B}\,\mathrm{pc}^{-2}} = 10^{-0.4(\mu_B - M_{\odot,B} - 21.572)}. \tag{9}$$

Using $\mu_{e,B} = 22$ mag arcsec^{-2} and $M_{\odot,B} = 5.31$, we obtain

$$I_{e,B} = 10^{-0.4(22-5.31-21.572)}\,L_{\odot,B}\,\mathrm{pc}^{-2}. \tag{10}$$

Therefore,

$$\boxed{I_{e,B} \simeq 89.7\,L_{\odot,B}\,\mathrm{pc}^{-2}}. \tag{11}$$

(c) For an exponential profile, the effective radius and scale length are related by

$$R_e = 1.678R_d \Rightarrow R_d = \frac{R_e}{1.678}. \tag{12}$$

Using the small-angle approximation, the physical effective radius is

$$R_e = d\theta, \tag{13}$$

where

$$\theta = \frac{10''}{206265} = 4.848 \times 10^{-5}\ \mathrm{rad}. \tag{14}$$

Since $d = 5$ Mpc $= 5 \times 10^6$ pc, we obtain

$$R_e = (5 \times 10^6)(4.848 \times 10^{-5})\ \mathrm{pc} \simeq 242.4\ \mathrm{pc}. \tag{15}$$

The scale length is then

$$R_d = \frac{242.4}{1.678} \simeq 144.5\ \mathrm{pc}. \tag{16}$$

At the effective radius, the surface brightness is

$$I_{e,B} = I_0 e^{-R_e/R_d}. \tag{17}$$

Thus, the central surface brightness is

$$I_0 = I_{e,B} e^{R_e/R_d} = I_{e,B} e^{1.678}. \tag{18}$$

Substituting $I_{e,B} = 89.7\ L_{\odot,B}\,\mathrm{pc}^{-2}$ gives

$$I_0 = 89.7 e^{1.678}\, L_{\odot,B}\,\mathrm{pc}^{-2} \simeq 480.5\, L_{\odot,B}\,\mathrm{pc}^{-2} \tag{19}$$

The total luminosity is obtained by integrating the surface brightness over the entire galaxy:

$$L_B = \int_0^\infty I(R)\, dS. \tag{20}$$

$$L_B = 2\pi \int_0^\infty I_0 e^{-R/R_d} R\, dR = \int_0^\infty R e^{-R/R_d}\, dR = R_d^2 = 2\pi I_0 R_d^2. \tag{21}$$

Substituting the numerical values gives

$$L_B = 2\pi (480.5)(144.5)^2\, L_{\odot,B} \simeq 6.30 \times 10^7\, L_{\odot,B}. \tag{22}$$

The absolute magnitude is related to the luminosity by

$$M_B = M_{\odot,B} - 2.5 \log_{10}\left(\frac{L_B}{L_{\odot,B}}\right). \tag{23}$$

Substituting the values,

$$M_B = 5.31 - 2.5 \log_{10}(6.30 \times 10^7). \tag{24}$$

Thus, before correcting for foreground extinction,

$$\boxed{M_B \simeq -14.19\ \mathrm{mag}}. \tag{25}$$

(d) The hydrogen column density is related to the visual extinction through the following empirical relation

$$N_H \simeq 1.8 \times 10^{21} A_V\,\mathrm{cm}^{-2}\,\mathrm{mag}^{-1} \Rightarrow A_V = \frac{N_H}{1.8 \times 10^{21}}\,\mathrm{mag}. \tag{26}$$

Substituting $N_H = 4 \times 10^{20}\ \mathrm{cm}^{-2}$, we obtain

$$A_V = \frac{4 \times 10^{20}}{1.8 \times 10^{21}} \simeq 0.222\ \mathrm{mag}. \tag{27}$$

Using the extinction parameter

$$R_V = \frac{A_V}{A_B - A_V} \Rightarrow A_B - A_V = \frac{A_V}{R_V} \Rightarrow A_B = A_V + \frac{A_V}{R_V} \tag{28}$$

Using $R_V = 3.1$, we obtain

$$A_B = 0.222 + \frac{0.222}{3.1}. \tag{29}$$

Hence,

$$\boxed{A_B \simeq 0.294 \text{ mag}}. \tag{30}$$

The extinction-corrected absolute magnitude is

$$M_{B,\text{corrected}} = M_B - A_B. \tag{31}$$

Substituting the previous result gives

$$M_{B,\text{corrected}} = -14.19 - 0.294. \tag{32}$$

Therefore,

$$\boxed{M_{B,\text{corrected}} \simeq -14.48 \text{ mag}}. \tag{33}$$

The Milky Way's B-band absolute magnitude is approximately $M_{B,\text{MW}} \simeq -20.5$ mag, which is lower due to more young stars compared to the given galaxy (consistent with a dwarf galaxy, although the B-band luminosity alone does not determine its total stellar mass).

3. **Mass Distribution in an Early-Type Galaxy**

Consider a massive early-type (elliptical) galaxy having an effective radius $R_e = 4$ kpc. The galaxy has been observed with a spectrograph that resolves individual absorption lines in the stellar spectrum. The lines are broadened due to internal stellar kinematics dominated by random motions rather than rotation. The equivalent width of the CaII line at 8542.09 Å (brightest line of the Calcium triplet) is 15 Å, where the equivalent width (EW) is defined as

$$\text{EW} \equiv \int_{\text{line}} \frac{F_{\lambda,\text{cont}}(\lambda) - F_\lambda(\lambda)}{F_{\lambda,\text{cont}}(\lambda)}\, d\lambda, \tag{34}$$

where F_λ is the measured flux at wavelength λ, $F_{\lambda,\text{cont}}$ is the continuum flux, and the integral is over the line.

(a) Assume that the line can be well described by a Gaussian function:

$$F(\lambda) = F_0 \left[1 - e^{-(\lambda-\lambda_0)^2/(2\sigma^2)}\right], \tag{35}$$

where F_0 is the continuum (assumed constant), λ_0 is the central wavelength, and σ the dispersion. Show that the equivalent width can be written as

$$\text{EW} = \sqrt{2\pi}\,\sigma. \tag{36}$$

(b) Taking $\lambda_0 = 8560.00$ Å, find the value of σ in km s^{-1}.

(c) Assume that the above obtained value of σ is a good measure of the inner velocity dispersion of the stars in the galaxy, σ_0, and use the virial theorem to obtain the virial mass $M_{\rm vir}$. Use a standard virial coefficient $k_{\rm vir} = 5$.

(d) Assuming that the galaxy contains hot gas in hydrostatic equilibrium and spherical symmetry, the total mass as a function of radius is:

$$M(r) = -\frac{k_B T r}{\mu m_p G}\left(\frac{d\ln\rho_{\rm gas}}{d\ln r} + \frac{d\ln T}{d\ln r}\right), \tag{37}$$

where $\rho_{\rm gas}$ and T are the gas density and temperature, and μ is the mean molecular weight. The density is well described by a β-model:

$$n(r) = n_0\left[1 + \left(\frac{r}{r_c}\right)^2\right]^{-3\beta/2}, \tag{38}$$

with $n_0 = 0.1$ cm^{-3}, $r_c = 20$ kpc, $\beta = 2/3$, and $T = 10^7$ K (constant). Use the above to calculate the total mass of the galaxy within $r = 200$ kpc. Discuss the difference from the result obtained in 3.

Tip 1: For $r \gg r_c$, you can use the asymptotic form of the β-model.

Tip 2: For a fully ionized gas, use $\mu \simeq 0.6$.

Solution:

(a) Using $F_{\lambda,\rm cont} = F_0$ and the Gaussian profile,

$$\begin{aligned}\mathrm{EW} &= \int_{-\infty}^{+\infty} \frac{F_0 - F_0\left[1 - e^{-(\lambda-\lambda_0)^2/(2\sigma^2)}\right]}{F_0}\, d\lambda \\ &= \int_{-\infty}^{+\infty} e^{-(\lambda-\lambda_0)^2/(2\sigma^2)}\, d\lambda.\end{aligned} \tag{39}$$

Let $x = (\lambda - \lambda_0)/\sigma$, so that $d\lambda = \sigma\, dx$. Using the Gaussian integral,

$$\begin{aligned}\mathrm{EW} &= \sigma\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx \\ &\Rightarrow \boxed{\mathrm{EW} = \sqrt{2\pi}\,\sigma}.\end{aligned} \tag{40}$$

(b) From part (a),

$$\begin{aligned}\mathrm{EW} &= \sqrt{2\pi}\,\sigma_\lambda \\ &\Rightarrow \sigma_\lambda = \frac{15}{\sqrt{2\pi}}\ \text{Å} \simeq 5.98\ \text{Å}.\end{aligned} \tag{41}$$

Using the non-relativistic Doppler relation,

$$\frac{\sigma_0}{c} = \frac{\sigma_\lambda}{\lambda_0} \Rightarrow \sigma_0 = \frac{c\,\sigma_\lambda}{\lambda_0} = \frac{(2.998\times10^5)(5.98)}{8560.00}\ \mathrm{km\,s^{-1}} \tag{42}$$

Therefore,

$$\boxed{\sigma_0 \simeq 209.6\ \mathrm{km\,s^{-1}}}. \tag{43}$$

(c) Using the virial mass estimator,

$$M_{\rm vir} = \frac{k_{\rm vir}\sigma_0^2 R_e}{G}. \tag{44}$$

Substituting $k_{\rm vir} = 5$, $R_e = 4$ kpc, and $G = 4.301 \times 10^{-6}\ \text{kpc}\,(\text{km\,s}^{-1})^2\,M_\odot^{-1}$,

$$\begin{aligned} M_{\rm vir} &= \frac{5(209.6)^2(4)}{4.301\times10^{-6}}\,M_\odot \\ &\Rightarrow \boxed{M_{\rm vir} \simeq 2.04\times10^{11}\,M_\odot}. \end{aligned} \tag{45}$$

(d) Since $\rho_{\rm gas} \propto n$ and T is constant,

$$\begin{aligned} \frac{d\ln T}{d\ln r} &= 0, \\ \ln n(r) &= \ln n_0 - \frac{3\beta}{2}\ln\left[1+\left(\frac{r}{r_c}\right)^2\right] \\ &\Rightarrow \frac{d\ln n}{d\ln r} = -\frac{3\beta r^2}{r_c^2+r^2}. \end{aligned} \tag{46}$$

Therefore,

$$M(r) = \frac{k_B T r}{\mu m_p G}\frac{3\beta r^2}{r_c^2+r^2}. \tag{47}$$

For $r \gg r_c$, this reduces to

$$M(r) \simeq \frac{3\beta k_B T r}{\mu m_p G} \overset{\beta=2/3}{\Rightarrow} M(r) \simeq \frac{2k_B T r}{\mu m_p G}. \tag{48}$$

At $r = 200$ kpc, with $r_c = 20$ kpc, we use the full expression:

$$M(200\text{ kpc}) = \frac{k_B T(200\text{ kpc})}{\mu m_p G}\frac{2(200)^2}{20^2+200^2}. \tag{49}$$

Using

$$\frac{k_B T}{\mu m_p} = \frac{(1.381\times10^{-23})(10^7)}{(0.6)(1.673\times10^{-27})} \simeq 1.37\times10^5\ (\text{km\,s}^{-1})^2, \tag{50}$$

we obtain

$$\begin{aligned} M(200\text{ kpc}) &= \frac{(1.37\times10^5)(200)}{4.301\times10^{-6}}\frac{80000}{40400}\,M_\odot \\ &\Rightarrow \boxed{M(200\text{ kpc}) \simeq 1.27\times10^{13}\,M_\odot}. \end{aligned} \tag{51}$$

Comparing with part (c), the hydrostatic mass is much larger because it measures the total mass within 200 kpc, including the extended dark matter halo, whereas the virial estimate uses stellar kinematics and the effective radius of 4 kpc. The two estimates therefore probe different spatial scales and are not directly equivalent.

4. **Two-Point Correlation Function in 3D and 2D**

The two-point correlation function $\xi(r)$ measures the excess probability (relative to random) of finding a galaxy at distance r from another galaxy:

$$dP = n(r)dV = n_0[1 + \xi(r)]dV, \tag{52}$$

where $n_0 \approx 0.03h^3$ Mpc^{-3} is the average number density for galaxies with $M_* > 10^9 M_\odot$.

(a) The number of galaxies within radius r is

$$N(r) = \int_0^r 4\pi r'^2 n(r')dr'. \tag{53}$$

For small distances, $\xi(r)$ follows a power law:

$$\xi(r) = \left(\frac{r}{r_0}\right)^{-\gamma}, \tag{54}$$

where $r_0 \approx 5h^{-1}$ Mpc and $\gamma \approx 1.8$. Derive a formula for $N(r)$.

(b) Determine within which of the following distances a galaxy has, on average, 5 neighbors: $r_5 = 0.3$ Mpc, 1.3 Mpc, or 2.0 Mpc.

(c) If a population had $\gamma > 1.8$, would it be more or less clustered? Give an example of a selection criterion that could produce a galaxy sample with higher γ.

(d) In observations, one may not have the 3D information but only a 2D projection of the galaxy density field. In that case, the surface density as a function of the projected radius R can be written:

$$\Sigma(R) = 2\int_R^{R_{\max}} \frac{n(r)r\,dr}{\sqrt{r^2 - R^2}} = \Sigma_0[1 + w(R)], \tag{55}$$

where $R_{\max}$ is a large distance that can be considered ∞ for the purpose of this integration, Σ_0 is the average surface density of galaxies and $w(R)$ is the projected two-point correlation function. Note: when sky coordinates (angles) are used, this is called the two-point angular correlation function. Show that if $\xi(r)$ is a power law as in Eq. (11), then $w(R)$ is also a power law with slope $1 - \gamma$.

Tip: The integral

$$\int_1^\infty \frac{x^{1-\gamma}}{\sqrt{x^2 - 1}}\,dx$$

is of order unity for $1 < \gamma < 3$.

Solution:

(a) Substituting into the expression for $N(r)$,

$$\begin{aligned} N(r) &= 4\pi \int_0^r r'^2 n_0 \left[1 + \left(\frac{r'}{r_0}\right)^{-\gamma}\right] dr' \\ &= 4\pi n_0 \left[\int_0^r r'^2\,dr' + r_0^\gamma \int_0^r r'^{2-\gamma}\,dr'\right] \\ &= 4\pi n_0 \left[\frac{r^3}{3} + \frac{r_0^\gamma r^{3-\gamma}}{3-\gamma}\right]. \end{aligned} \tag{56}$$

Therefore,

$$\boxed{N(r) = 4\pi n_0 \left[\frac{r^3}{3} + \frac{r_0^\gamma r^{3-\gamma}}{3-\gamma}\right]}. \tag{57}$$

(b) Using $n_0 = 0.03h^3$ Mpc^{-3}, $r_0 = 5h^{-1}$ Mpc, and $\gamma = 1.8$,

$$N(r) = 4\pi(0.03)\left[\frac{h^3 r^3}{3} + \frac{5^{1.8} h^{1.2} r^{1.2}}{1.2}\right]. \tag{58}$$

Adopting $h = 0.7$, we evaluate the expected number of neighbors at each radius:

$$\begin{aligned} N(0.3 \text{ Mpc}) &\simeq 0.88, \\ N(1.3 \text{ Mpc}) &\simeq 5.17, \\ N(2.0 \text{ Mpc}) &\simeq 8.86. \end{aligned} \tag{59}$$

Thus, the radius at which a galaxy has approximately five neighbors is

$$\boxed{r_5 = 1.3 \text{ Mpc}}. \tag{60}$$

(c) The correlation function is

$$\xi(r) = \left(\frac{r}{r_0}\right)^{-\gamma}. \tag{61}$$

A larger γ means that the correlation function decreases more rapidly with distance. At fixed r_0, this implies stronger clustering on small scales. Therefore, a population with $\gamma > 1.8$ is more strongly clustered on small scales. For example, selecting massive, luminous red elliptical galaxies can produce a more strongly clustered sample than selecting galaxies of all types, because massive red galaxies tend to reside in denser environments.

(d) The projected surface density is

$$\Sigma(R) = 2\int_R^\infty \frac{n(r)r}{\sqrt{r^2 - R^2}}\, dr. \tag{62}$$

Using $n(r) = n_0[1 + \xi(r)]$ and separating the uncorrelated and correlated contributions, the excess projected density is proportional to

$$w(R) \propto \int_R^\infty \frac{\xi(r)r}{\sqrt{r^2 - R^2}}\, dr. \tag{63}$$

Substituting the power-law correlation function gives

$$w(R) \propto r_0^\gamma \int_R^\infty \frac{r^{1-\gamma}}{\sqrt{r^2 - R^2}}\, dr. \tag{64}$$

Let $x = r/R$, so that $r = Rx$ and $dr = R\,dx$. Then

$$w(R) \propto r_0^\gamma R^{1-\gamma} \int_1^\infty \frac{x^{1-\gamma}}{\sqrt{x^2 - 1}}\, dx. \tag{65}$$

For $1 < \gamma < 3$, the integral is a constant independent of R. Therefore,

$$w(R) \propto r_0^{\gamma} R^{1-\gamma}. \tag{66}$$

Hence, the projected correlation function is also a power law:

$$\boxed{w(R) \propto R^{1-\gamma}}. \tag{67}$$

Its slope is therefore $1 - \gamma$.

5. **Cosmic Time and Luminosity Distance in Simplified Universes**

For a large fraction of cosmic time, the evolution of the Universe is dominated by the matter component. Such a Universe behaves like an Einstein–de Sitter (EdS) model, characterized by $\Omega_{\mathrm{m},0} = 1$ and $k = 0$.

(a) Using the first Friedmann equation,

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2, \tag{68}$$

where a and $\dot{a}$ are the scale factor and its time derivative, respectively, and ρ is the matter density, show that the relation between cosmic time t and redshift z can be written as:

$$t(z) = \frac{2}{3H_0}\frac{1}{(1+z)^{3/2}}, \tag{69}$$

where $H_0 = 70$ km s^{-1} Mpc^{-1} is the Hubble constant.

(b) Using the Hubble parameter appropriate for an EdS Universe, find an expression for the luminosity distance d_L. Remember that it is related to the comoving distance x by $d_L = (1+z)x$.

(c) Consider now a Universe dominated by the cosmological constant, $\Omega_{\Lambda,0} = 1$ and $k = 0$. Find the corresponding expression for the luminosity distance d_L in this case.

(d) Consider two galaxies, one at $z = 0.15$ and another at $z = 5$. Use Eq. (14) to calculate the cosmic time t_c at those redshifts in an EdS Universe, then use the other results obtained above to calculate the luminosity distances of these galaxies in both an EdS and a Λ-dominated Universe.

Solution:

(a) For an EdS Universe, matter density scales as

$$\rho = \rho_0 a^{-3}, \qquad \rho_0 = \frac{3H_0^2}{8\pi G}.$$

Substituting into the Friedmann equation,

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_0 a^{-1} = H_0^2 a^{-1} \Rightarrow \dot{a} = H_0 a^{-1/2}.$$

Therefore,

$$a^{1/2}da = H_0 dt \Rightarrow \frac{2}{3}a^{3/2} = H_0 t.$$

Using $a = (1+z)^{-1}$,

$$\boxed{t(z) = \frac{2}{3H_0(1+z)^{3/2}}}.$$

(b) For an EdS Universe,

$$H(z) = H_0(1+z)^{3/2}.$$

The comoving distance is

$$x(z) = c\int_0^z \frac{dz'}{H(z')} = \frac{c}{H_0}\int_0^z (1+z')^{-3/2}dz'$$

$$= \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right].$$

Since $d_L = (1+z)x$,

$$\boxed{d_L^{\mathrm{EdS}}(z) = \frac{2c}{H_0}\left[(1+z) - \sqrt{1+z}\right]}.$$

(c) For a flat, cosmological-constant-dominated Universe,

$$H(z) = H_0, \qquad x(z) = c\int_0^z \frac{dz'}{H_0} = \frac{cz}{H_0}.$$

Hence,

$$\boxed{d_L^{\Lambda}(z) = \frac{c}{H_0}z(1+z)}.$$

(d) Using $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $c = 3.00 \times 10^5\ \mathrm{km\,s^{-1}}$,

$$\frac{c}{H_0} = 4285.71\ \mathrm{Mpc}, \qquad t_0 = \frac{2}{3H_0} \approx 9.31\ \mathrm{Gyr}.$$

The cosmic time and luminosity distances are:

	$z = 0.15$	$z = 5$
t_c (Gyr)	$9.31/1.15^{3/2} = 7.54$	$9.31/6^{3/2} = 0.634$
d_L^{EdS} (Mpc)	$4285.71 \cdot 2(1.15 - \sqrt{1.15}) = 669$	$4285.71 \cdot 2(6 - \sqrt{6}) = 30,680$
d_L^{Λ} (Mpc)	$4285.71(0.15)(1.15) = 739$	$4285.71(5)(6) = 128,571$

6. **Mass–Size Relation at $z = 2.5$**

The stellar mass–size relation describes the correlation between the stellar mass ($M_\star$) of galaxies and the effective radius (R_e) of their stellar distribution. Using the recent work of Ward et al. [2024], the relation for massive star-forming galaxies at $z \approx 2.5$ is given by:

$$R_e = A \left(\frac{M_\star}{5 \times 10^{10}\ M_\odot} \right)^\alpha \text{ kpc}, \tag{70}$$

where $A \simeq 3.02$ is the intercept and $\alpha = 0.15$ the slope.

(a) Consider the role of surface brightness. How much fainter is a galaxy at $z = 2.5$ compared to $z = 0$, accounting only for cosmological surface brightness dimming? Are there other factors that could make the observed surface brightness higher or lower at $z = 2.5$ compared to nearby galaxies? Mention one and explain in which direction it acts.

(b) To explore the origin of the mass–size relation for star-forming galaxies, assume that:

1. all galaxies are well described by exponential profiles, with surface brightness

$$I(R) = I_0 \exp(-R/R_d), \tag{71}$$

where I_0 is the central surface brightness and R_d the scale radius;
2. they all have the same central surface brightness I_0; and
3. the conversion between surface brightness and surface density (Σ) is done via a constant mass-to-light ratio Y, such that $\Sigma_0 = Y I_0$.

Show that under these assumptions, the expected slope of a relation like Eq. (15) is 0.5. Discuss why this predicted slope is steeper than the observed value ($\alpha = 0.15$), and which of the assumptions above might be unrealistic.

Solution:

(a) Cosmological surface brightness dimming follows

$$I_{\text{obs}} = \frac{I_{\text{emit}}}{(1+z)^4}.$$

At $z = 2.5$,

$$\frac{I_{\text{obs}}}{I_{\text{emit}}} = \frac{1}{3.5^4} = \frac{1}{150.06}.$$

Therefore, the galaxy appears approximately

$$\boxed{150 \text{ times fainter}}$$

Other factors can also affect the observed surface brightness, such as young stars in star-forming galaxies since they can increase their intrinsic luminosity per unit area, making them brighter and partially offsetting cosmological dimming.

(b) For an exponential surface brightness profile,

$$I(R) = I_0 e^{-R/R_d},$$

the total luminosity is

$$L = 2\pi \int_0^\infty I(R) R \, dR = 2\pi I_0 R_d^2.$$

Using a constant mass-to-light ratio Y,

$$M_\star = YL = 2\pi Y I_0 R_d^2.$$

Since I_0 and Y are constant,

$$M_\star \propto R_d^2 \Rightarrow R_d \propto M_\star^{1/2}.$$

For an exponential profile, the effective radius is

$$R_e = 1.678 R_d \Rightarrow R_e \propto M_\star^{1/2}.$$

Thus, the predicted slope is

$$\boxed{\alpha = 0.5}.$$

The observed slope, $\alpha = 0.15$, is shallower because one or more assumptions may be unrealistic:

- **Constant central surface brightness:** I_0 may vary with stellar mass, weakening the predicted scaling.
- **Constant mass-to-light ratio:** Y may vary with stellar mass due to differences in stellar populations, ages, and dust attenuation.
- **Exponential profiles:** Real galaxies may deviate from perfect exponential profiles.

7. Linear and Non-linear Perturbations

Primordial fluctuations observed in the Cosmic Microwave Background (CMB) at recombination ($z_{\rm rec} \approx 1000$) correspond to a baryonic density contrast $\delta_b \sim 10^{-5}$. This is too small to form the observed structures (galaxies and clusters), implying that there were larger fluctuations ($\delta_{\rm DM} \sim 10^{-3}$) in the dark matter component.

(a) Explain why the above implies that:

1. dark matter must be non-baryonic, and
2. dark matter must be "cold," i.e. non-relativistic at decoupling, assuming dark matter particles are thermal relics.

(b) In an EdS Universe, the growth factor for linear perturbations is $D(t) = a(t)$. If a dark matter perturbation has $\delta_{\rm DM} = 0.003$ at $z_{\rm rec}$, what will $\delta_{\rm DM}$ be at $z = 100$? Estimate the baryonic perturbation δ_b at the same redshift.

(c) Now consider that a perturbation becomes non-linear and virializes into a dark matter halo at $z = 20$. Using the spherical collapse model, the virial mass is

$$M_{\rm vir} = \frac{4\pi}{3}\Delta_c \rho_{\rm crit} r_{\rm vir}^3, \tag{72}$$

where $\Delta_c = 178$ is the critical overdensity for virialization in an EdS Universe, and $\rho_{\rm crit}(z) = \frac{3H^2(z)}{8\pi G}$. Substitute $\rho_{\rm crit}$ and the expression for $H(z)$ in an EdS Universe to compute the virial radius at $z = 2.5$ for a halo with $M_{\rm vir} = 10^{12}\ M_\odot$.

Solution:

(a) 1. **Dark matter must be non-baryonic.**

Baryons are coupled to photons before recombination, so photon pressure prevents baryonic perturbations from growing significantly. Since $\delta_b \sim 10^{-5} < \delta_{\rm DM} \sim 10^{-3}$, the dark matter perturbations must have grown independently of the baryonic-photon fluid. Therefore, dark matter must be predominantly non-baryonic.

2. **Dark matter must be cold.**

Thermal relics that are relativistic have large thermal velocities and free-stream out of small-scale density perturbations, suppressing structure formation. Cold dark matter is non-relativistic at decoupling, allowing perturbations to survive and grow into galaxies and clusters.

(b) In an EdS Universe,

$$D(z) \propto a(z) = \frac{1}{1+z}.$$

Therefore,

$$\delta(z) = \delta(z_{\rm rec})\frac{1+z_{\rm rec}}{1+z}.$$

At $z = 100$,

$$\delta_{\rm DM}(100) = 0.003\frac{1001}{101} \approx \boxed{0.0297}.$$

Similarly, assuming baryonic perturbations grow at the same linear rate after recombination,

$$\delta_b(100) = 10^{-5}\frac{1001}{101} \approx \boxed{9.91 \times 10^{-5}}.$$

(c) The virial mass is

$$M_{\rm vir} = \frac{4\pi}{3}\Delta_c \rho_{\rm crit}(z) r_{\rm vir}^3, \qquad \rho_{\rm crit}(z) = \frac{3H^2(z)}{8\pi G}.$$

Substituting,

$$M_{\rm vir} = \frac{4\pi}{3}\Delta_c \frac{3H^2(z)}{8\pi G} r_{\rm vir}^3 = \frac{\Delta_c H^2(z)}{2G} r_{\rm vir}^3.$$

Hence,

$$r_{\rm vir} = \left[\frac{2GM_{\rm vir}}{\Delta_c H^2(z)}\right]^{1/3}.$$

For an EdS Universe,

$$H(z) = H_0(1+z)^{3/2},$$

so

$$\boxed{r_{\rm vir} = \left[\frac{2GM_{\rm vir}}{\Delta_c H_0^2 (1+z)^3}\right]^{1/3}}.$$

Using $M_{\rm vir} = 10^{12} M_\odot$, $\Delta_c = 178$, $z = 2.5$, $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$, and $G = 4.301 \times 10^{-9}\ \mathrm{Mpc\,(km\,s^{-1})^2 M_\odot^{-1}}$,

$$r_{\rm vir} = \left[\frac{2(4.301\times10^{-9})(10^{12})}{178(70)^2(3.5)^3}\right]^{1/3}\ \mathrm{Mpc}.$$

Therefore,

$$\boxed{r_{\rm vir} \approx 0.0635\ \mathrm{Mpc} = 63.5\ \mathrm{kpc}}.$$