Practice session: Linear Algebra (For Physics)

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Give motivations and/or derivations for your answers.

1. **Question 1**

Check whether in each case the given vectors form a basis in the corresponding vector space. Justify your response.

(a) $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

If yes, give the transition matrices from the V basis to the E basis, and from E basis to V basis. Also write the coordinate vector xE given in basis E as a coordinate vector xV in basis V.

$x_E = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

(b) $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 6 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

If yes, give the transition matrices from the V basis to the E basis, and from E basis to V basis. Also write the coordinate vector xE given in basis E as a coordinate vector xV in basis V.

$x_E = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}$

Solution: Calculate the determinant:
- If the determinant is zero, the vectors are linearly dependent.
- If the determinant is nonzero, the vectors are linearly independent.

(a)

$$\det = 0$$

Since the determinant is zero, the three vectors are linearly dependent and cannot form a basis.

(b)

$$\det = 15$$

Since the determinant is nonzero, the three vectors are linearly independent and form a basis.

The inverse of the matrix is: . $\begin{bmatrix}1/5\\2/5\\0\\0\end{bmatrix}\begin{bmatrix}2/5\\-1/5\\0\\0\end{bmatrix}\begin{bmatrix}0\\0\\1/3\\2\end{bmatrix}\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}$

so $\mathrm{x}_V is :.$ $\begin{bmatrix}0\\0\\1/3\\-2\end{bmatrix}$

2. **Question 2**

Determine whether the following sets of vectors are linearly independent:

(a) $\{1, x^2, x^2 - 1\} \in P_3$

(b) $\{\sinh(x), \cosh(x)\} \in C^2[0, 1]$

(c) Given the vectors:

$$p(x) = 2a, \quad q(x) = 4x + 2, \quad r(x) = (a - 2)x^2,$$

with $p(x), q(x), r(x) \in P_3$ and a to be determined, find for which values of a the three vectors are linearly independent and therefore form a basis of P_3.

Solution: Calculate the Wronskian to determine if the vectors are linearly independent.
- If the Wronskian is nonzero, the vectors are linearly independent.
- If the Wronskian is zero, further analysis is needed.

(a)

$$W(1, x^2, x^2 - 1) = 0$$

can express $x^2 - 1$ as a linear combination of 1 and x^2, so the set is linearly dependent.

(b)

$$W(\sinh(x), \cosh(x)) = -1$$

Since the Wronskian is nonzero, the functions are linearly independent.

(c)

$$W(p(x), q(x), r(x)) \neq 0 \quad \text{for} \quad 16a(a - 2) \neq 0$$

So, the vectors are linearly independent for $a \neq 0$ and $a \neq 2$.

Question 3

The coupled system of 2 differential equations is given by:
$x_1'(t) = 2x_1(t) + 3x_2(t)$
$x_2'(t) = -2x_1(t) + 6x_2(t)$
with boundary conditions $x_1(0) = 0.5\sqrt{2}$, $x_2(0) = -1$.

(a) Compute the (complex) eigenvalues and eigenvectors of this system.

(b) Give the general real-valued solution of this system of differential equations.

(c) Give the real-valued solution, x(t), of the initial-value problem.

Solution:

(a) Solution question 3a. Given the system represented by the matrix:

$$A = \begin{bmatrix} 2 & 3 \\ -2 & 6 \end{bmatrix}$$

We need to find the eigenvalues λ by using:

$$\det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ -2 & 6-\lambda \end{bmatrix}$$

$$\lambda_1 = 4 + i\sqrt{2}, \quad \lambda_2 = 4 - i\sqrt{2}$$

Eigenvector for $\lambda_1 = 4 + i\sqrt{2}$:

$$\begin{bmatrix} 2-(4+i\sqrt{2}) & 3 \\ -2 & 6-(4+i\sqrt{2}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\mathbf{v}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1-(i\sqrt{2})/2 \\ 1 \end{bmatrix}$$

Eigenvector for $\lambda_2 = 4 - i\sqrt{2}$:

$$\mathbf{v}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1+(i\sqrt{2})/2 \\ 1 \end{bmatrix}$$

(b) To obtain the real-valued solution, we write the general solution in terms of real and imaginary parts. Let:

$$\mathbf{v}_1 = \mathbf{Re}(\mathbf{v}) + i\mathbf{Im}(\mathbf{v})$$

The real-valued general solution is:

$$\mathbf{x}(t) = e^{4t}\left[c_1\left(\mathbf{Re}(\mathbf{v})\cos(\sqrt{2}t) - \mathbf{Im}(\mathbf{v})\sin(\sqrt{2}t)\right) + c_2\left(\mathbf{Re}(\mathbf{v})\sin(\sqrt{2}t) + \mathbf{Im}(\mathbf{v})\cos(\sqrt{2}t)\right)\right]$$

Expanding the components:

$$\mathbf{x}(t) = e^{4t}\left[c_1\begin{bmatrix}\cos(\sqrt{2}t) + \sqrt{2}/2\sin(\sqrt{2}t)\\ \cos(\sqrt{2}t))\end{bmatrix} + c_2\begin{bmatrix}\sin(\sqrt{2}t) - \sqrt{2}/2\cos(\sqrt{2}t)\\ \sin(\sqrt{2}t)\end{bmatrix}\right]$$

(c) where c_1, c_2 are constants that can be determined by the initial conditions, c_1=-1 c_2=-1-$\sqrt{2}$

$$\mathbf{x}(t) = e^{4t}\left[\begin{bmatrix}-\cos(\sqrt{2}t) - \sqrt{2}/2\sin(\sqrt{2}t)\\ -\cos(\sqrt{2}t))\end{bmatrix} - (1+\sqrt{2})\begin{bmatrix}\sin(\sqrt{2}t) - \sqrt{2}/2\cos(\sqrt{2}t)\\ \sin(\sqrt{2}t)\end{bmatrix}\right]$$

Question 4

Given the vectors:

$$\mathbf{v} = \begin{bmatrix}\gamma\\2\\5\end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix}4\\-1\\3\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}2\\3\\-2\end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix}-1\\\delta\\4\end{bmatrix}$$

with the parameters γ and δ to be determined.

(a) For which value(s) of γ and δ is the vector $\mathbf{v}$ in Span$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$?

(b) For which value(s) of δ are **a, b and c** linear dependent? Express **a** as a linear combination of **b** and **c** in this case.

(c) Using the value(s) of δ from part (b), give the 3 unit vectors in the direction of the vectors **a, b and c**.

(d) For which value(s) of δ are the vectors **b** and **c** orthogonal?

Solution: We need to determine for which values of γ and δ the vector $\mathbf{v}$ is in the span of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. This means that there exist scalars x, y, and z such that:

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{v}. \tag{1}$$

Substituting the given vectors:

$$x\begin{bmatrix}4\\-1\\3\end{bmatrix}+y\begin{bmatrix}2\\3\\-2\end{bmatrix}+z\begin{bmatrix}-1\\\delta\\4\end{bmatrix}=\begin{bmatrix}\gamma\\2\\5\end{bmatrix}. \tag{2}$$

We get the augmented matrix:

$$\begin{bmatrix}4 & 2 & -1 & |\gamma\\-1 & 3 & \delta & |2\\3 & -2 & 4 & |5\end{bmatrix}. \tag{3}$$

$$\begin{bmatrix}4 & 2 & -1 & |\gamma\\0 & 14 & -1+4\delta & |\gamma+8\\0 & 0 & \frac{7-2\delta}{2} & |\gamma-142\end{bmatrix}. \tag{4}$$

Conclusion: v will be in Span {a, b, c} if:
1) $\delta \neq \frac{7}{2}$ (one solution)

OR

2) If $\delta = \frac{7}{2}$**and** $\gamma = 142$ (the last row is all 0's, there is a free variable and so there are infinite solutions.)

b) For this we need to calculate

$$det[abc] = det\begin{bmatrix}4 & 2 & -1\\-1 & 3 & \delta\\3 & -2 & 4\end{bmatrix}. \tag{5}$$

and find for which value(s) $\delta = 0$, which is the condition for the vectors to be linearly dependent. The determinant turns out to be equal to $14\delta + 63$ therefore if $\delta = \frac{-63}{14}$ or $\frac{-9}{2}$ the vectors are linearly dependent.

To express **a** as a linear combination of **b** and **c** we need to find the values of λ and μ that verify the equation:

$$a = \lambda b + \mu c \tag{6}$$

and solve the corresponding system of equations for one of the values of δ we get:

$$a = 19/16b + 7/3c \tag{7}$$

c) Taking $\delta = -9/2$, the three vectors are:

$$\mathbf{a}=\begin{bmatrix}4\\-1\\3\end{bmatrix},\quad \mathbf{b}=\begin{bmatrix}2\\3\\-2\end{bmatrix},\quad \mathbf{c}=\begin{bmatrix}-1\\-\frac{9}{2}\\4\end{bmatrix}.$$

The moduli of the three vectors are:

$$|\mathbf{a}| = \sqrt{26}, \quad |\mathbf{b}| = \sqrt{17}, \quad |\mathbf{c}| = \sqrt{\frac{149}{4}}.$$

The unit vectors are:

$$\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{26}}\begin{bmatrix}4\\-1\\3\end{bmatrix}, \quad \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{1}{\sqrt{17}}\begin{bmatrix}2\\3\\-2\end{bmatrix}, \quad \frac{\mathbf{c}}{|\mathbf{c}|} = \sqrt{\frac{4}{149}}\begin{bmatrix}-1\\-\frac{9}{2}\\4\end{bmatrix}.$$

d) Two vectors are orthogonal if and only if their dot product is equal to zero.

$$a^{\mathbf{T}}b = 2(1) + 3\delta + (-2)4 = 0. \tag{8}$$

So the two vectors will be orthogonal if $\delta = \frac{10}{3}$.

Question 5

Let P_3 be the vector space consisting of all polynomials p with real coefficients of degree less than 3. For each transformation $L : P_3 \to P_3$ defined below, show that the transformation is linear and find the matrix representation of L with respect to the basis $A = \{x^2, x, 1\}$.
Also, apply the transformation using the matrix to the polynomial $p(x) = ax^2 + bx + c$ written as the coordinate vector with respect to the basis A, and show that this is the same as calculating the derivatives/integrals of the polynomial $p(x)$ in the usual way.

(a)

$$L(p) = \left(0.5\frac{d}{dx} + 4\frac{d^2}{dx^2} - 2\right)p(x)$$

(b)

$$L(p) = e^x \frac{d}{dx}\left(e^{-x}p(x)\right)$$

Solution:

(a)

$$L(x^2) = -2x^2 + x + 8 = -2(x^2) + 1(x) + 8(1)$$
$$L(x) = -2x + \frac{1}{2} = 0(x^2) - 2(x) + \frac{1}{2}(1)$$
$$L(1) = -2 = 0(x^2) + 0(x) - 2(1)$$

So, B is:

$$B = \begin{bmatrix}-2\\1\\8\end{bmatrix}, \quad \begin{bmatrix}0\\-2\\\frac{1}{2}\end{bmatrix}, \quad \begin{bmatrix}0\\0\\-2\end{bmatrix}$$

So, with x_A, we have:

$$x_A = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Then, $B \times x_A$ is:

$$Bx_A = \begin{bmatrix} -2a \\ a - 2b \\ 8a + \frac{1}{2}b - 2c \end{bmatrix}$$

Now, apply the transformation L:

$$L(p) = \left(0.5\frac{d}{dx} + 4\frac{d^2}{dx^2} - 2\right)(ax^2 + bx + c)$$

$$= -2ax^2 - 2bx - 2c + \frac{1}{2}b + ax + 8a$$

(b)

$$L(p) = -p(x) + \frac{d}{dx}p(x)$$

$$L(x^2) = -x^2 + 2x = -1(x^2) + 2(x) + 0(1)$$

$$L(x) = -x + 1 = 0(x^2) - 1(x) + 1(1)$$

$$L(1) = -1 = 0(x^2) - 0(x) - 1(1)$$

Then B is:

$$B = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Then, $B \times x_A$ is:

$$Bx_A = \begin{bmatrix} -a \\ 2a - b \\ b - c \end{bmatrix}$$

Now, apply the transformation L:

$$L(p) = e^x \frac{d}{dx}\left(e^{-x}(ax^2 + bx + c)\right)$$

$$= -(ax^2 + bx + c) + \frac{d}{dx}(ax^2 + bx + c)$$

$$= -ax^2 + (2a - b)x + (b - c)$$