Practice session: Linear Algebra (For Physics)

Kapteyn Learning Community Sirius A
Give motivations and/or derivations for your answers.

1. **Question 1**

   Check whether in each case the given vectors form a basis in the corresponding vector space. Justify your response.

   (a) $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

   If yes, give the transition matrices from the V basis to the E basis, and from E basis to V basis. Also write the coordinate vector xE given in basis E as a coordinate vector xV in basis V.

   $x_E = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$

   (b) $v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 6 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

   If yes, give the transition matrices from the V basis to the E basis, and from E basis to V basis. Also write the coordinate vector xE given in basis E as a coordinate vector xV in basis V.

   $x_E = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix}$

2. **Question 2**

   Determine whether the following sets of vectors are linearly independent:

   (a) $\{1, x^2, x^2 - 1\} \in P_3$

   (b) $\{\sinh(x), \cosh(x)\} \in C^2[0, 1]$

   (c) Given the vectors:

   $$p(x) = 2a, \quad q(x) = 4x + 2, \quad r(x) = (a - 2)x^2,$$

   with $p(x), q(x), r(x) \in P_3$ and a to be determined, find for which values of a the three vectors are linearly independent and therefore form a basis of P_3.

Question 3

The coupled system of 2 differential equations is given by:
$x_1'(t) = 2x_1(t) + 3x_2(t)$
$x_2'(t) = -2x_1(t) + 6x_2(t)$
with boundary conditions $x_1(0) = 0.5\sqrt{2}$, $x_2(0) = -1$.

(a) Compute the (complex) eigenvalues and eigenvectors of this system.

(b) Give the general real-valued solution of this system of differential equations.

(c) Give the real-valued solution, x(t), of the initial-value problem.

Question 4

Given the vectors:

$$\mathbf{v} = \begin{bmatrix} \gamma \\ 2 \\ 5 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -1 \\ \delta \\ 4 \end{bmatrix}$$

with the parameters γ and δ to be determined.

(a) For which value(s) of γ and δ is the vector $\mathbf{v}$ in Span$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$?

(b) For which value(s) of δ are **a, b and c** linear dependent? Express **a** as a linear combination of **b** and **c** in this case.

(c) Using the value(s) of δ from part (b), give the 3 unit vectors in the direction of the vectors **a, b and c**.

(d) For which value(s) of δ are the vectors **b** and **c** orthogonal?

Question 5

Let P_3 be the vector space consisting of all polynomials p with real coefficients of degree less than 3. For each transformation $L : P_3 \to P_3$ defined below, show that the transformation is linear and find the matrix representation of L with respect to the basis $A = \{x^2, x, 1\}$.
Also, apply the transformation using the matrix to the polynomial $p(x) = ax^2 + bx + c$ written as the coordinate vector with respect to the basis A, and show that this is the same as calculating the derivatives/integrals of the polynomial $p(x)$ in the usual way.

(a)

$$L(p) = \left(0.5\frac{d}{dx} + 4\frac{d^2}{dx^2} - 2\right) p(x)$$

(b)

$$L(p) = e^x \frac{d}{dx}\left(e^{-x} p(x)\right)$$