

Cauchy - Riemann eq.:  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$   $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$   
 where  $f(z) = u(x, y) + i v(x, y)$ ,  $z = x + iy$

Laplacian:  $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \rightarrow$  helps find HARMONIC func  $u$ , if 2<sup>nd</sup> order part. der. are cont. and Laplac. satisf.

Exponential form:  $e^{i\theta} = \cos \theta + i \sin \theta$   
 let  $r = |z|$  and  $\theta = \text{Arg } z \rightarrow z = r e^{i\theta} = r(\cos \theta + i \sin \theta)$

De Moivre:  $(e^{i\theta})^n = e^{ni\theta} \Rightarrow (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

BASIC FORMULAS:

$$|z| = \sqrt{x^2 + y^2}$$

$$\bar{z} = x - iy = r e^{-i\theta}$$

$$z \cdot \bar{z} = |z|^2$$

Binomial exp.  $\Rightarrow$  coeff. given by Pascal's triangle:

$$(a+b)^4 = 1 \cdot a^4 + 4 \cdot a^3 b + 6 \cdot a^2 b^2 + 4 \cdot a b^3 + 1 \cdot b^4$$

$$(a+b)^5 = 1 \cdot a^5 + 5 \cdot a^4 b + 10 \cdot a^3 b^2 + 10 \cdot a^2 b^3 + 5 \cdot a b^4 + 1 \cdot b^5$$

$$\frac{1}{i} = -i$$

$$\frac{z}{w} = \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \dots = \frac{ac+bd}{c^2+d^2} + i \frac{bc-ad}{c^2+d^2}$$

Residues:  $\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z-z_0)^m f(z))$   $m = \text{order of pole}$

Cauchy's integral formula:  $\int_C \frac{f(z)}{(z-z_0)^m} dz = 2\pi i \cdot \frac{1}{(m-1)!} \cdot f^{(m-1)}(z_0)$

$\hookrightarrow$  identify pole  $z_0$  of order  $m$ , and check if  $|z_0| < |z_0|$

- if contour  $C$  traversed in the negative dir. (clockwise)  $\rightarrow \times (-1)$
- if contour  $C$  traversed  $n$  times  $\rightarrow \times n$

Series expansions:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^m}{m!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \rightarrow \text{ODD terms}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \rightarrow \text{EVEN terms}$$

if  $|z| < 1 \rightarrow \frac{1}{1-z} = \sum_{j=0}^{\infty} (z)^j$

Taylor:  $f(x) = f(a) + \dots + (x-a)^n \frac{f^{(n)}(a)}{n!}$

$\rightarrow$  Laurent series: Taylor, but w.  $a_j$  terms in the coeff.  
 $\hookrightarrow$  differs for same func. based on annulus

Singularities  $\rightarrow$  poles, removable, essential:

1. POLES:  $f(z) = g(z)/h(z)$

$z_0$  is a pole if  $h(z_0) = 0$ , but  $g(z_0) \neq 0$

order  $m \rightarrow h(z) = (z - z_0)^m$

OBS: If  $h$  is of the form  $(z^2 - k) = (z - ki)(z + ki)$   
 $\Rightarrow z_0 = \pm ki$  is a pole of order 1, not 2

2. REMOVABLE SING.:

$\hookrightarrow$  if  $f(z)$  can be rewritten (usually in a series exp.) in such a way that the singularity disappears

3. ESSENTIAL SING.:

$\hookrightarrow$  if the Laurent exp. of  $f(z)$  has infinitely many  $\frac{1}{z}$  terms

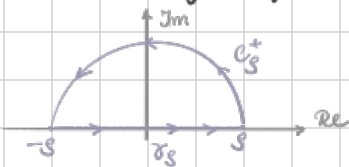
Method of residues for integrals:

given a func.  $f(z)$ , you want to compute p.v.  $\int_{-\infty}^{\infty} f(x) dx$

~ step 1: find the poles of  $f(z)$  and their order

~ step 2: use the residue formula to get  $\text{Res}(f; z_0)$

~ step 3: contour of integration:  $\Gamma_S = \gamma_S + C_S^+$  where  $\begin{cases} \gamma_S = \{x: -S \leq x \leq S\} \\ C_S^+ = \{S e^{it}: 0 \leq t \leq \pi\} \end{cases}$



Cauchy's residue th.:  $\int_{\Gamma} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f; z_k)$

~ step 4: Check which poles are included in the contour. Only those count towards the sum in Cauchy's formula.

~ step 5:  $\int_{\Gamma_S} f(z) dz = \int_{\gamma_S} f(z) dz + \int_{C_S^+} f(z) dz = \int_{-S}^S f(x) dx + \int_{C_S^+} f(z) dz$

principal value: p.v.  $\int_{-\infty}^{\infty} f(x) dx = \lim_{S \rightarrow \infty} \int_{-S}^S f(x) dx$

- find  $\int_{\gamma_S} f(z) dz$  using Cauchy

-  $\int_{C_S^+} f(z) dz = 0$ , probably due to the lemma:  $\deg(\text{num}) \geq \deg(\text{denom.})$   
 $\Rightarrow \lim_{S \rightarrow \infty} \int_{C_S^+} f(z) dz = 0$

$\Rightarrow$  p.v.  $\int_{-\infty}^{\infty} f(x) dx = \text{Cauchy's form. result}$

If  $C$  is any circle centered around  $z_0$  and  $n \in \mathbb{N}$ :

$$\oint_C (z - z_0)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

$$\int_{\Gamma} \frac{1}{z-a} dz = \begin{cases} 0, & \text{if } a \text{ lies out of } \Gamma \\ 2\pi i, & \text{if } a \text{ lies in } \Gamma \end{cases}$$

