

Lecture 6:

• Integral over a smooth directed curve γ :

$$\begin{aligned}\int_{\gamma} f(z) dz &:= \int_a^b f(z(t)) z'(t) dt \\ &= \int_a^b \operatorname{Re}(f(z(t)) z'(t)) dt + i \int_a^b \operatorname{Im}(\dots) dt\end{aligned}$$

$$\left. \begin{array}{l} z(t), a \leq t \leq b \\ w(s), c \leq s \leq d \end{array} \right\} \Rightarrow \phi: [c, d] \rightarrow [a, b] \text{ such that: } z(\phi(s)) = w(s)$$

• Integral over a contour Γ :

$$\int_{\Gamma} f(z) dz := \sum_{k=1}^n \int_{\gamma_k} f(z) dz \quad \nearrow f \text{ is cont. on } \Gamma$$

OBS: $\oint_C \frac{1}{z} dz = 2\pi i$ when C is the circle enclosing the origin (bc. of discontinuity), and $= 0$ in other cases

— general: $\oint_C \frac{1}{z-a} dz = 2\pi i$ when C goes once around a counterclockwise

OBS: If the func. is analytic (smooth, no singularities) inside and on the curve, then the integral around a closed path is always 0.

• Antiderivatives:

$$\left. \begin{array}{l} f(z) \text{ continuous in domain } D \\ F'(z) = f(z), \forall z \in D \end{array} \right\} \Rightarrow \int_{\Gamma} f(z) dz = F(z_i) - F(z_f)$$

$$\int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt = \int_a^b \frac{d}{dt} [F(z(t))] dt = F(z(b)) - F(z(a))$$

→ equivalent statements:

- f has an antiderivative in D
- if Γ is a loop, then $\int_{\Gamma} f(z) dz = 0$
- if Γ_1 and Γ_2 have the same initial and terminal point, then $\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz$

Lecture 4:

- check slides for vector fields

Line integrals:

GREEN'S theorem:

- D simply connected
 - $V_1, 2$ cont. diff.
 - Γ pos. oriented simple closed cont. in D
- for D' enclosed by Γ :

$$\int_{\Gamma} (V_1 dx + V_2 dy) = \iint_{D'} \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) dx dy$$

CAUCHY'S theorem:

th. If f is analytic in a simply connected domain D and Γ is any loop in D , then: $\int_{\Gamma} f(z) dz = 0$
↳ f has an antider. and cont. integr. of f are path indep.

th. Assume: → D is a simply connected domain
→ f analytic in D
→ Γ a simple, closed, + oriented contour in D
if z_0 lies in the interior of Γ :

$$f(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - z_0} dz \quad \nearrow z_0 - \text{point of discontinuity}$$

Lecture 8:

Sequences: $c_0, c_1, c_2, c_3, c_4, \dots$

the sequence (c_n) converges to a limit L if for all $\varepsilon > 0$ there $(\exists) N \in \mathbb{N}$ such that $|c_n - L| < \varepsilon$ whenever $n \geq N$

Partial Sums:

series $\Rightarrow \infty$ sum of complex nr.

partial sum of first $n+1$ terms: $S_n = \sum_{j=0}^n c_j$

S_n converges to a nr. S as $n \rightarrow \infty \Rightarrow$ convergent

COMPARISON test: for some $k \in \mathbb{N}$ we have

- $|c_j| \leq K_j$ for all $j \geq k$
- $\sum_{j=0}^{\infty} K_j$ converges

$\rightarrow \sum_{j=0}^{\infty} c_j$ converges too

RATIO test: assume nr. c_j satisfy $\lim_{j \rightarrow \infty} |c_{j+1}/c_j| = L$

then the series $\sum_{j=0}^{\infty} c_j$ conv. if $L < 1$ and div. if $L > 1$

Taylor Series: $\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!} \cdot (z-z_0)^j$ T.S. for f around z_0

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \\ \sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \\ \cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \end{aligned}$$

$\rightarrow$ taking $e^{iy} = (\dots) = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \dots\right) + i \left(y - \frac{y^3}{3!} + \dots\right) = \cos(y) + i \sin(y)$

th. f analytic in disk $D = \{z \in \mathbb{C} : |z - z_0| < R\}$
 $\rightarrow$ T.S. around z_0 converges to $f(z)$ for all $z \in D$

Products:

$$\sum_{j=0}^{\infty} a_j (z-z_0)^j \quad \sum_{j=0}^{\infty} b_j (z-z_0)^j \quad \longrightarrow \quad \sum_{j=0}^{\infty} c_j (z-z_0)^j \quad \text{with} \quad c_j = \sum_{k=0}^j a_k b_{j-k}$$

Power series

$$\sum_{j=0}^{\infty} a_j (z-z_0)^j = a_0 + a_1 (z-z_0) + \underbrace{a_2 (z-z_0)^2}_{\text{coeff.}} + \dots$$

Convergence:

th. For any power series there $(\exists)$ a nr. $0 \leq R \leq \infty$ depending only on the coeff. a_j such that

1. the series converges when $|z-z_0| < R$
2. the series diverges when $|z-z_0| > R$

- in the disk $|z-z_0| < R$ the sum of the pow. ser. is an analytic func.

Relation to T.S.:

th. If $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ conv. to $f(z)$ in D around z_0 , then necessarily:

$$a_j = \frac{f^{(j)}(z_0)}{j!}$$

- for $C = \{z \in \mathbb{C} : |z-z_0| = r\}$ with $r < R$: $a_j = \frac{1}{2\pi i} \int_C f^{(j)}(z) / (z-z_0)^{j+1} dz$

Lecture 9:

Laurent Series:

ANNULUS = region of complex plane enclosed by 2 concentric circles

th.: f analytic in annulus $r < |z - z_0| < R \rightarrow f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}$
where both series converge in the annulus.

- for $C = \{z : |z - z_0| = \rho\}$ w. $r < \rho < R$ we have: $a_j = \frac{1}{2\pi i} \int_C f(z) / (z - z_0)^{j+1} dz \quad j \in \mathbb{Z}$

- see slide 9 for series examples

GOAL: expand smth. (e.g. $\frac{1}{z-1}$) as a series $\rightarrow$ either powers of z or $\frac{1}{z}$ using:

$$\frac{1}{1-x} = \sum_{j=0}^{\infty} x^j \rightarrow \text{geometric series formula} \rightarrow \text{VALID only when } |x| < 1 \rightarrow \text{so it converges!}$$

$$\text{trick: } |x| > 1 \Rightarrow \frac{1}{1-x} = \frac{1}{x} \cdot \frac{1}{\frac{1}{x} - 1} = -\frac{1}{x} \cdot \frac{1}{1 - \frac{1}{x}} = -\frac{1}{x} \cdot \sum_{j=0}^{\infty} \left(\frac{1}{x}\right)^j$$

OBS: $|z - a| < b \Rightarrow$ inside circle centered at a , of radius b

$$r < |z - a| < R \rightarrow$$


• Zeros :

Assume f analytic at $z_0 \Rightarrow z_0$ is called a zero of order m if :

- $f^{(k)}(z_0) = 0$ for $k = 0, \dots, m-1$
- $f^{(m)}(z_0) \neq 0$

• Singularities :

Assume in annulus $0 < |z - z_0| < R$ we have : $f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j$

- z_0 is called :
- **REMOVABLE** sing. if $a_j = 0$ for all $j < 0$
 - **POLE of ORDER m** if $a_j = 0$ for all $j < -m$ and $a_{-m} \neq 0$
 - **ESSENTIAL** sing. if $a_j \neq 0$ for inf. many $j \neq 0$

Lecture 10:

Residues:

Assume f has a Laurent series exp.: $f(z) = \sum_{j=-\infty}^{\infty} a_j (z-z_0)^j$ in some annulus centered at z_0 .

The **RESIDUE** at z_0 is defined as: $\text{Res}(f; z_0) = a_{-1} \rightarrow$ residue = coeff. of the $\frac{1}{z}$ term in its Laurent series expansion

CLAIM: Assume — f has a pole of order 1 at z_0

— g is analytic at z_0

Then we have: $\text{Res}(f \cdot g; z_0) = g(z_0) \text{Res}(f; z_0)$

Assume — f is analytic at z_0

— $f(z_0) = 0$ but $f'(z_0) \neq 0$

Then we have: $\text{Res}(f'; z_0) = f'(z_0)$

COROLLARY: Consider $f(z) = p(z)/q(z)$ where — p and q analytic at z_0

— q has a zero of order 1 at z_0 and $p(z_0) \neq 0$

Then we have: $\text{Res}(f; z_0) = p(z_0)/q'(z_0)$

THEOREM: If f has a pole of order m at z_0 , then:

$$\text{Res}(f; z_0) = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$$

Cauchy's Residue Theorem:

Assume: — Γ is a positively oriented simple closed contour

— f is analytic inside and on Γ except at $z_1, \dots, z_m$ inside Γ

Then we have that:

$$\int_{\Gamma} f(z) dz = 2\pi i \sum_{j=1}^m \text{Res}(f; z_j)$$

Trigonometric Integrals: goal: compute integrals of the form $\int_0^{2\pi} U(\cos \theta, \sin \theta) d\theta$

key idea: Find a function $f(z)$ such that: $\int_{|z|=1} f(z) dz = \int_0^{2\pi} U(\cos \theta, \sin \theta) d\theta$

$$\rightarrow \int_0^{2\pi} f(e^{i\theta}) \overbrace{ie^{i\theta}}^{dz} d\theta = \int_0^{2\pi} U\left(\frac{e^{i\theta} + e^{-i\theta}}{2}, \frac{e^{i\theta} - e^{-i\theta}}{2i}\right) d\theta$$

$$f(z) = U\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right) \cdot \frac{1}{iz}$$

$$z = e^{i\theta} \rightarrow 1/z = e^{-i\theta}$$

Lecture 1.1:

• Integral over a half axis:

DEF.:

If f continuous on $[0, \infty)$, then:

$$\int_0^{\infty} f(x) dx := \lim_{b \rightarrow \infty} \int_0^b f(x) dx$$

If f continuous on $(-\infty, 0]$, then:

$$\int_{-\infty}^0 f(x) dx := \lim_{a \rightarrow -\infty} \int_a^0 f(x) dx$$

If f cont. on the whole real line, then:

$$\begin{aligned} \int_{-\infty}^{\infty} f(x) dx &:= \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx \\ &:= \lim_{a \rightarrow -\infty} \int_a^0 f(x) dx + \lim_{b \rightarrow \infty} \int_0^b f(x) dx \\ &:= \lim_{s \rightarrow \infty} \int_{-s}^s f(x) dx \end{aligned}$$

PRINCIPAL VALUE:

If f is cont. on the whole real line, then we def. the CAUCHY PRINC. VAL. of the integr. as:

$$\text{p.v. } \int_{-\infty}^{\infty} f(x) dx := \lim_{s \rightarrow \infty} \int_{-s}^s f(x) dx$$