1 ADVANCED ELECTRODYNAMICS HW 5

REFERENCE FRAMES:

$B_x = ?$, $B_y = f(y)\cos(Kx)$, $B_z = 0$

WHERE $f(-y) = -f(y)$

(a) SINCE THERE ARE NO CHARGES AND CURRENT WE HAVE:

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = 0 \end{cases}$$

$$\left\{\begin{array}{ll} \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} = 0 & \Big] \operatorname{div}\vec{B} = 0 \\ \dfrac{\partial B_y}{\partial z} = 0 & \\ \dfrac{\partial B_x}{\partial z} = 0 & \operatorname{curl}\vec{B} = 0 \\ \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = 0 & \end{array}\right.$$

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y} = -f'(y)\cos(Kx)$$

$$\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = -Kf(y)\sin(Kx)$$

SINCE $\int \cos Kx \, dx = \frac{1}{K}\sin(Kx)$ + CONSTANT, WE ARE LOOKING FOR AN EVEN FUNCTION $f(y)$ FOR WHICH:

$$\frac{1}{K}f'(y) = K\int f(y)\,dy$$

THAT IS $\cosh(Ky)$

AND:

$$\boxed{\begin{aligned} B_y &= B_0\cos(Kx)\cosh(Ky) \\ B_x &= -B_0\sin(Kx)\sinh(Ky) \end{aligned}}$$

SINCE $\dfrac{\partial B_x}{\partial x} = -f'(y)\cos Kx = -K\sinh(Ky)\cos(Kx)$

PROOF:

$$\left.\begin{aligned} \frac{\partial B_x}{\partial x} &= -B_0K\cos(Kx)\sinh(Ky) \\ -\frac{\partial B_y}{\partial y} &= -B_0K\cos(Kx)\sinh(Ky) \end{aligned}\right\} \rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

$$\left.\begin{aligned} \frac{\partial B_y}{\partial x} &= -B_0K\sin(Kx)\cosh(Ky) \\ \frac{\partial B_x}{\partial y} &= -B_0K\sin(Kx)\cosh(Ky) \end{aligned}\right\} \rightarrow \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = 0$$

b) the frame S' moves at $\vec{v} = v\hat{x}$ relative to S.
The initial velocity of the electron is $\vec{v}$ w.r.t. S, so it is initially at rest in S'.
In S' the fields are:

$E_x' = E_x = 0$, $E_y' = \gamma(E_y - vB_z) = 0$, $E_z' = \gamma(E_z + vB_y) = \gamma v B_y$

$B_x' = B_x$, $B_y' = \gamma(B_y + \frac{v}{c^2}E_z) = \gamma B_y$, $B_z' = \gamma(B_z - \frac{v}{c^2}E_y) = \gamma B_z$

SINCE THE ELECTRON IS AT REST IN S', THE FORCE ON IT IS:

$\vec{F}' = -e\vec{E}'$

THAT IS:

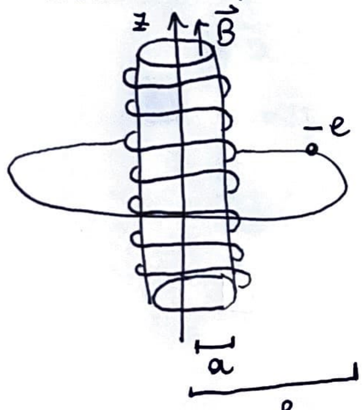
$$\begin{cases} F_x' = 0 \\ F_y' = 0 \\ F_z' = -ev\gamma B_y \end{cases}$$

and applying the second law $\vec{F} = m\vec{a}$

$$\begin{cases} m\ddot{x}' = 0 \\ m\ddot{y}' = 0 \\ m\ddot{z}' = -ev\gamma B_0 \cos(kx)\sinh(ky) \end{cases}$$

These are the equations of motion.
The electron has an acceleration in z direction and then it radiates.

2



$\vec{A} = -\vec{r} \times \frac{\vec{B}}{2}$ IS THE VECTOR POTENTIAL

$A_\rho = A_z = 0$

$A_\phi = \rho \frac{B}{2}$

a) The Hamiltonian is:

$H = (-i\hbar\vec{\nabla} + e\vec{A})^2/2m + V(\rho))$

The magnetic flux Φ_B through the solenoid is:

$\Phi_B = \int_S \vec{B}\cdot d\vec{S} = B\int dS = \pi a^2 B$

and $\vec{A} = \frac{\Phi_B}{2\pi\rho}\hat{\phi}$, $V(\rho) = 0$ (no charge in the solenoid)

$H = \frac{1}{2m}(-\hbar^2\nabla^2 + e^2A^2 - 2ie\hbar\vec{A}\cdot\vec{\nabla})$

where in cylindrical coordinate:

$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$

$\vec{A}\cdot\vec{\nabla} = \frac{1}{\rho}\left(\frac{\Phi_B}{2\pi\rho}\right)\frac{\partial}{\partial\phi}$

$H = \frac{1}{2m}\left[-\frac{\hbar^2}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{\hbar^2}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \left(\frac{e\Phi_B}{2\pi\rho}\right)^2 - \frac{ie\hbar\Phi_B}{\pi\rho^2}\frac{\partial}{\partial\phi}\right]$

IS THE HAMILTONIAN IN CYLINDRICAL COOR

b) If we can consider the torus as a ring of radius ρ, we can fix ρ as a constant and, since $\theta = \frac{\pi}{2}$, there is only dependence on the Azimuthal angle ϕ.

The Hamiltonian becomes:

$$H = \frac{1}{2m}\left[-\frac{\hbar^2}{\rho^2}\frac{d^2}{d\phi^2} + \left(\frac{e\Phi_B}{2\pi\rho}\right)^2 - i\frac{\hbar e \Phi_B}{\pi\rho^2}\frac{d}{d\phi}\right]$$

where now ρ is a constant

since $L_z = -i\hbar\frac{\partial}{\partial\phi}$ we can write:

$$H = \frac{1}{2m\rho^2}\left(L_z + \frac{e\Phi_B}{2\pi}\right)^2$$

To find the energy eigenvalues we have to write the time independent Schrödinger equation:

$$H\psi = E\psi \qquad \text{where } \psi = \psi(\phi)$$

$$-\frac{\hbar^2}{\rho^2}\frac{d^2\psi}{d\phi^2} - \frac{i\hbar e\Phi_B}{\pi\rho^2}\frac{d\psi}{d\phi} + \left[\left(\frac{e\Phi_B}{2\pi\rho}\right)^2 - 2mE\right]\psi = 0$$

$$\frac{d^2\psi}{d\phi^2} + \frac{\rho^2}{\hbar^2}i\frac{\hbar e}{\pi\rho^2}\Phi_B\frac{d\psi}{d\phi} + \left[2mE\frac{\rho^2}{\hbar^2} - \frac{\rho^2}{\hbar^2}\frac{e^2\Phi_B^2}{4\pi^2\rho^2}\right]\psi = 0$$

$$\frac{d^2\psi}{d\phi^2} + i\frac{e\Phi_B}{\pi\hbar}\frac{d\psi}{d\phi} + \left[\frac{2m\rho^2}{\hbar^2}E - \frac{e^2\Phi_B^2}{4\pi\hbar^2}\right]\psi = 0$$

let $P = -\frac{e\Phi_B}{2\pi\hbar}$, $Q = 2m\frac{\rho^2}{\hbar^2}E - P^2$

with the substitutions:

$$\frac{d^2\psi}{d\phi^2} - 2iP\frac{d\psi}{d\phi} + Q\psi = 0$$

The solutions are $\psi = Be^{i\lambda\phi}$:

$$\frac{d\psi}{d\phi} = i\lambda B e^{i\lambda\phi}$$

$$\frac{d^2\psi}{d\phi^2} = -\lambda^2 B e^{i\lambda\phi}$$

substituting:

$$-\lambda^2 B e^{i\lambda\phi} + 2\lambda P B e^{i\lambda\phi} + QBe^{i\lambda\phi} = 0$$

$$\lambda^2 - 2\lambda P - Q = 0 \rightarrow \lambda = P \pm \sqrt{P^2+Q} = P \pm \sqrt{\frac{2m\rho^2 E}{\hbar^2}}$$

for the continuity of the wavefunction:

$$\psi(0) = \psi(2\pi) \rightarrow e^0 = e^{i\lambda\cdot 2\pi} \Rightarrow \lambda \text{ is an integer}$$

then:

$$\lambda = -\frac{e\Phi_B}{2\pi\hbar} \pm \frac{\rho}{\hbar}\sqrt{2mE} = n$$

$$\pm\frac{\rho}{\hbar}\sqrt{2mE_n} = n + \frac{e\Phi_B}{2\pi\hbar}$$

$$2mE_n = \frac{\hbar^2}{\rho^2}\left(n + \frac{e\Phi_B}{2\pi\hbar}\right)^2$$

and the energy eigenvalues are:

$$\boxed{E_n = \frac{\hbar^2}{2m\rho^2}\left(n + \frac{e\Phi_B}{2\pi\hbar}\right)^2} \qquad n = 0, \pm 1, \pm 2, \dots$$

3

evaluation of:

$$\mu_B = \frac{e\hbar}{2m_e}, \quad \Phi = \frac{h}{2e}, \quad Ry = \frac{m_e e^4}{8\varepsilon_0^2 h^2}$$

$\hbar c \approx 6.6 \times 10^{-16}\, eV \cdot 3 \times 10^{17}\, nm \approx 200\, eV \cdot nm$

$m_e c^2 \approx 9.1 \times 10^{-31}\, kg \cdot c^2 \approx 500\, k\, eV$

$\alpha = \frac{e^2}{4\pi \varepsilon_0 \hbar c} \approx \frac{1}{137}$ is the fine structure constant

$V = \frac{kg\, m^2}{A\, s^3}, \quad T = \frac{kg}{A s^2}$

a) The Bohr magnetron is:

$$\mu_B = \frac{e\hbar}{2m_e} = \frac{ec(\hbar c)}{2(m_e c^2)} = \frac{eV \cdot c \cdot (\hbar c)}{2V\,(m_e c^2)} = \frac{eV \cdot c \cdot (\hbar c)}{2 \cdot \frac{kg\, m^2}{A s^3}\,(m_e c^2)}$$

$$\mu_B \approx \frac{eV \cdot 3 \cdot 10^8\, m s^{-1} \cdot 200\, eV \cdot 10^{-9}\, m}{2 \cdot kg\, m^2 A^{-1} s^{-3}\; 500\cdot 000\; eV} \approx 6 \times 10^{-5}\, eV\, \frac{A s^2}{kg}$$

$$\boxed{\mu_B \approx 6 \times 10^{-5}\, eV/T}$$

b) The magnetic flux quantum is:

$$\Phi = \frac{h}{2e} = \frac{2\pi \hbar V}{2eV} = \frac{2\pi\,(\hbar c)}{2eV\, c}\, V$$

$$= \frac{2\pi \cdot 200\, eV \cdot 10^{-9}\, m}{2eV \cdot 3 \cdot 10^8\, m s^{-1}}\, V \approx 2 \cdot 10^{-15}\, V \cdot s = 2 \cdot 10^{-15}\, Wb$$

$$\boxed{\Phi \approx 2 \times 10^{-15}\, Wb}$$

c) The rydberg energy is:

$$Ry = \frac{m_e e^4}{8\varepsilon_0^2 h^2} = \frac{m_e e^4 c^2}{8\varepsilon_0^2 4\pi^2 \hbar^2 c^2} = \frac{(m_e c^2)}{2}\left(\frac{e^2}{4\pi\varepsilon_0 \hbar c}\right)^2$$

$$= \frac{m_e c^2}{2} \cdot \alpha^2 \simeq \frac{500\cdot 000}{2}\, eV \cdot \frac{1}{137^2} \simeq 13.3\, eV$$

$$\boxed{Ry \approx 13.3\, eV}$$