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ADVANCED ELECRODYNAMICS $\boxed{\text{HW 4}}$

$\boxed{1}$ (a) $L = \frac{1}{2} m v_\mu v^\mu + q v_\mu A^\mu$ is the Lagrangian

where v^μ is the four-vector of the charged particle and A^μ is the electromagnetic four potential.

The Euler - Lagrange equations are:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial v_\alpha}\right) = \frac{\partial L}{\partial x_\alpha}$$

we have ; $L = \frac{1}{2} m v_\mu(\tau) v^\mu(\tau) + q v_\mu(\tau) A^\mu(x)$

and:

$$\frac{d}{d\tau}\left(m v_\alpha + q A^\alpha\right) = q v^\mu \frac{\partial A^\mu}{\partial x_\alpha}$$

$$\frac{d}{d\tau}\left(m v_\alpha\right) + \frac{d}{d\tau}\left(q A^\alpha\right) = q v^\mu \frac{\partial A^\mu}{\partial x_\alpha}$$

with $\frac{dA^\alpha}{d\tau} = \frac{\partial A^\alpha}{\partial x^\mu} \cdot \frac{dx^\mu}{d\tau} = \frac{\partial A^\alpha}{\partial x^\mu} v^\mu$

then

$$\frac{d}{d\tau}\left(m v_\alpha\right) + q \frac{\partial A^\alpha}{\partial x^\mu} v^\mu = q v^\mu \frac{\partial A^\mu}{\partial x_\alpha}$$

and

$$\frac{d}{d\tau}\left(m v_\alpha\right) = q v^\mu \left[\frac{\partial A^\mu}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x^\mu}\right]$$

$$\frac{d}{d\tau}\left(m v_\alpha\right) = q v^\mu F_{\mu\alpha}$$ where $F_{\mu\alpha} = \left[\frac{\partial A^\mu}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x^\mu}\right]$

(b) The canonical momentum is:

$$p^\alpha = \frac{\partial L}{\partial v_\alpha} = m v^\alpha + q A^\mu$$

and the hamiltonian

$$H = P_\alpha v^\alpha - L = m v_\alpha v^\alpha + q A^\mu - \frac{1}{2} m v_\mu v^\mu - q A^\mu$$

$$H = mc^2 - \frac{1}{2} mc^2 = \frac{1}{2} mc^2$$

since $v_\alpha v^\alpha = v_\mu v^\mu = c^2$

$\boxed{2}$

$$\mathcal{L} = -\frac{1}{2\mu_0} \partial_\alpha A_\beta \partial^\alpha A^\beta + J_\alpha A^\alpha$$

(a) we have:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = J^\mu, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = -\frac{1}{\mu_0} \partial^\nu A^\mu$$

and the Euler-Lagrangian equations:

$$\partial \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right) = \frac{\partial \mathcal{L}}{\partial A^\mu} \longrightarrow -\frac{1}{\mu_0} \partial^\nu \partial_\nu A^\mu = J^\mu$$

that is:

$$\partial_\nu \partial^\nu A^\mu + \mu_0 J^\mu = 0$$

That are the Maxwell equations in Lorentz gauge condition

$$\Box^2 A^\mu = -\mu_0 J^\mu$$

(b) The usual density is:

$$\mathcal{L}_0 = -\frac{1}{\mu_0} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu$$

$$\mathcal{L} - \mathcal{L}_0 = \frac{1}{\mu_0} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2\mu_0} \partial_\mu A_\nu \partial^\mu A^\nu$$

the term $F^{\mu\nu} F_{\mu\nu}$ is gauge invariant

$$\mathcal{L} - \mathcal{L}_0 = \frac{1}{2\mu_0} \partial_\mu A_\nu \partial^\nu A^\mu = \frac{1}{2\mu_0} \partial_\mu \left(A_\nu \partial^\nu A^\mu - A^\mu \partial^\nu A_\nu \right) + \frac{1}{2\mu_0} (\partial^\mu A_\mu)^2 =$$

$$\frac{1}{2\mu_0} \partial_\mu \left(A_\nu \partial^\nu A^\mu - A^\mu \partial^\nu A_\nu \right)$$

in the Lorentz gauge $\partial^\mu A_\mu = 0$

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The addition of the 4-divergence doesn't affect, since its integral gives only a surface term.

$\boxed{3}$ The corrected Coulomb law is:

$$\vec{F} = \frac{qQ}{4\pi\varepsilon_0}(1+\mu r)\frac{e^{-\mu^2}}{r^2}\hat{r}$$

a) The electric field is:

$$\vec{E} = \frac{\vec{F}}{q} = \frac{Q}{4\pi\varepsilon_0}(1+\mu r)\frac{e^{\mu r}}{r^2}\hat{r}$$

$$\rho(r) = Q\delta(r)$$

b) The Gauss law is

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} - \mu^2\phi$$